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ON THE DECOMPOSITION OF TENSOR PRODUCTS OF PRINCIPAL SERIES REPRESENTATIONS FOR REAL-RANK ONE SEMISIMPLE GROUPS

BY

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ABSTRACT. Let G be a connected semisimple real-rank one Lie group with finite center. It is shown that the decomposition of the tensor product of two representations from the principal series of G consists of two pieces, T_c and T_d, where T_c is a continuous direct sum with respect to Plancherel measure on $\hat{G}$ of representations from the principal series only, occurring with explicitly determined multiplicities, and T_d is a discrete sum of representations from the discrete series of G, occurring with multiplicities which are, for the present, undetermined.

I. Introduction. Let G be a connected semisimple Lie group with finite center. If P is a cuspidal parabolic subgroup of G and $P = MAN$ denotes a Langlands decomposition for P, we denote by $\hat{M}_d$ the square-integrable irreducible representations of M. For $\sigma \in \hat{M}_d$, $\tau \in \hat{A}$, $\sigma \times \tau$ is a representation of MAN via $(\sigma \times \tau)(man) = \sigma(m)\tau(a)$ and the family of representations

$$\{\pi(\sigma, \tau) = \operatorname{Ind}_P^G \sigma \times \tau\colon\ \sigma \in \hat{M}_d,\ \tau \in \hat{A}\}$$

is called the nondegenerate continuous series corresponding to P. In the case of a minimal parabolic subgroup, it is customary to say principal series. An important problem is that of decomposing the tensor product of two such representations into irreducibles. In §5, it is shown that this problem "reduces" to knowing how to decompose tensor products of representations from $\hat{M}_d$ and how to decompose $\operatorname{Ind}_{MA}^G \sigma \times \tau$ for all $(\sigma, \tau) \in \hat{M}_d \times \hat{A}$. One of the main goals of this paper is to show to what extent these last two problems can be answered when G has real-rank one, $P = MAN$ is a minimal parabolic subgroup, and we are decomposing the tensor product of two principal series representations.

The main result is that this decomposition consists of two pieces, T_c and

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T_d, where T_c is a continuous direct sum with respect to Plancherel measure on $\hat{G}$ of representations from the principal series only, occurring with explicitly determined multiplicities, and T_d is a discrete direct sum of representations from the discrete series of G, occurring with multiplicities which are, for the present, undetermined. Let $V = \theta N$, where θ is an appropriate Cartan involution on G, and S denote a suitably chosen cross-section for the action of MA on V. Then both the cardinality of S and the isotropy subgroup at $v_0 \in S$ under the action of MA (which is actually independent of $v_0 \in S$) play decisive roles in determining the multiplicity of a principal series representation occuring in T_c.

For real-rank one groups we shall show, by using Mackey's tensor product theorem, that the problem of decomposing the tensor product of two principal series representations reduces to knowing $\mathrm{Ind}_{MA}^G\ \sigma \times \tau$ for *all* $\sigma \times \tau \in (MA)^\wedge$. We then show (in fact for arbitrary rank) that $\mathrm{Ind}_{MA}^G\ \sigma \times \tau$ is independent of $\tau \in \hat{A}$ and hence it suffices to determine $\mathrm{Ind}_{MA}^G\ \sigma \times \tau$ for *almost all* $\sigma \times \tau \in (MA)^\wedge$. By applying a reciprocity theorem due to N. Anh, this amounts to determining the multiplicity of $\sigma \times \tau$ in the restriction of π to MA, $(\pi)_{MA}$, for almost all $\pi \in \hat{G}$.

If π is an irreducible representation from the principal series of G, then $(\pi)_{MA}$ is computed by applying Mackey's subgroup theorem to MAN and MA. It is here that explicit knowledge of V/MA is needed. If S denotes a suitable cross-section for this action, then S depends not only on the number of positive roots which are not simple roots but also on the dimensions of the root spaces.

For irreducible representations π of the discrete series of G we first give a new proof (one using Anh's reciprocity theorem) of the fact that there exists a $\delta \in \hat{K}$ such that π is contained in $\mathrm{Ind}_K^G\ \delta$. We then use Mackey's subgroup theorem to compute $(\mathrm{Ind}_K^G\ \delta)_{MA}$. From this it becomes clear that

$$(\pi)_{MA} \simeq \int_{MA}^{\oplus} n(\sigma, \tau, \pi)(\sigma \times \tau)\, d\mu_C(\sigma, \tau)$$

where μ_C is Plancherel measure on $(MA)^\wedge$ and $n(\sigma, \tau, \pi) \in \{0, 1, 2, \cdots, \infty\}$.

The problem of decomposing the tensor product of principal series representations has been considered for $\mathrm{SL}(2, \mathbf{C})$ by G. Mackey in [13] and M. A. Naĭmark in [14], for $\mathrm{SL}(n, \mathbf{C})$ by N. Anh in [1], and was completely solved for complex semisimple groups by F. Williams in his 1972 thesis [18]. For $\mathrm{SL}(2, \mathbf{R})$, the problem was completely solved by L. Pukanszky in [15]. We shall comment on these cases in §6 of this paper and show how the techniques developed in §5 not only can be used to give new proofs of these results but also can be used to give a complete solution to the problem of decomposing the tensor product of two (minimal) principal series representations of $G = \mathrm{SL}(n, \mathbf{R})$, $n \geqslant 3$.

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II. **The semisimple theory.** Let G be a connected semisimple Lie group with finite center and Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition, θ denote the corresponding Cartan involution, and $K =$ the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Let $B_\theta(X, Y) = -B(X, \theta Y)$ where B is the Killing form and $X, Y \in \mathfrak{g}$. Then B_θ is an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$ which makes $\mathfrak{g}$ into a real Hilbert space. Let $\mathfrak{a}$ be a maximal (abelian) subalgebra of $\mathfrak{p}$ and $\mathfrak{a}^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{R})$ its dual. For $\lambda \in \mathfrak{a}^*$, put $\mathfrak{g}_\lambda = \{X \in \mathfrak{g}\colon [H, X] = \lambda(H)X, \text{ for all } H \in \mathfrak{a}\}$. If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$, then λ is called a (restricted) root and $m_\lambda = \dim \mathfrak{g}_\lambda$ is called its multiplicity. For $\alpha \in \mathfrak{a}^*$, let $H_\alpha \in \mathfrak{a}$ be determined by $\alpha(H) = B_\theta(H, H_\alpha)$ for $H \in \mathfrak{a}$. Let $\mathfrak{a}'$ be the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$. The components of $\mathfrak{a}'$ are called Weyl chambers and if we fix a Weyl chamber $\mathfrak{a}^+$, a root α is called positive if it is positive on $\mathfrak{a}^+$. Let Δ (resp. Δ_+) denote the set of roots (resp. positive roots). A root $\alpha \in \Delta_+$ is called simple if it is not the sum of two positive roots. The simple roots form a basis for $\mathfrak{a}^*$. Put

$$\mathfrak{n} = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \qquad \mathfrak{v} = \theta\mathfrak{n} = \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha},$$

and let N, V and A denote the analytic subgroups of G with Lie algebras $\mathfrak{n}$, $\mathfrak{v}$ and $\mathfrak{a}$ respectively. Then $G = KAN$ is an Iwasawa decomposition and the dimension of A is called the real-rank of G. Let M (resp. M') denote the centralizer (resp. normalizer) of A in K. Then M is normal in M', both M, M' are closed, $W = M'/M$ is the (finite) Weyl group, and MAN is a (minimal parabolic) subgroup of G.

If $m'_1, \cdots, m'_w$ is a set of representatives of W and $P = MAN$, then we have the Bruhat decomposition $G = \bigcup_{i=1}^{w} Pm'_iP$. This union is disjoint and exactly one of the summands, viz., Pm^*P where $\mathrm{Ad}(m^*)\mathfrak{a}^+ = \mathfrak{a}^- = \{H\colon \alpha(H) < 0, \text{ for all } \alpha > 0\}$, is open in G. So $MANV$ has a complement of measure zero in G.

By a parabolic subgroup P of G is meant a closed subgroup of G such that

(i) if $\mathfrak{b} = LA(P)$, then P is the normalizer of $\mathfrak{b}$ in G, and

(ii) $\mathfrak{b}_{\mathbf{C}}$ contains a maximal solvable subalgebra of $\mathfrak{g}_{\mathbf{C}}$.

A parabolic subgroup P is called minimal if it is minimal among all parabolic subgroups of G. Let $N =$ the maximal unipotent normal subgroup of P, set $\Xi = P \cap \theta P$, and set $A =$ the maximal connected split (i.e., $\mathrm{Ad}(a)$ diagonizable over $\mathbf{R}$) abelian subgroup lying in the center of Ξ. Then Ξ is the centralizer of A in G. Let $X(\Xi) = \{\chi\colon \Xi \longrightarrow \mathbf{R}^*, \ \chi \text{ a continuous homomorphism}\}$. Set $M = \bigcap_{\chi \in X(\Xi)} \ker |\chi|$. Then M is reductive (i.e., $\mathfrak{m} = LA(M)$ is reductive) but not connected in general. Moreover, $\Xi = MA$ is a direct product and the map $(m, a, n) \longrightarrow man$ is an analytic diffeomorphism of $M \times A \times N$ onto P. This is called the Langlands decomposition for P. P is said to be cuspidal if M has a compact Cartan subgroup. For the results quoted above, we refer to [5] and [17].

Let $P = MAN$ be a cuspidal parabolic subgroup of G and $\hat{M}_d$ denote the square-integrable irreducible representations of M. Then for $\tau \in \hat{A}$, $\sigma \in \hat{M}_d$, $\sigma \times \tau$ is a representation of MAN via $(\sigma \times \tau)(man) = \sigma(m)\tau(a)$ and the family of representations $\{\pi(\sigma, \tau) = \mathrm{Ind}_P^G \sigma \times \tau\colon \sigma \in \hat{M}_d, \tau \in \hat{A}\}$ is called the non-degenerate continuous series corresponding to P. In the case of a minimal parabolic subgroup, it is customary to say principal series. It is known that almost all the $\pi(\sigma, \tau)$ are irreducible and, as in the case of a minimal parabolic subgroup of G, $MAN\bar{N}$ $(\bar{N} = \theta N)$ has a complement of measure zero in G. Two cuspidal parabolics P_1, P_2 are called associate if there exists $x \in G$ such that $xA_1x^{-1} = A_2$. Conjugating by x, we may assume that $P_1 = MAN_1$, $P_2 = MAN_2$. Finally, it can be shown that $\mathrm{Ind}_{P_1}^G \sigma \times \tau \simeq \mathrm{Ind}_{P_2}^G \sigma \times \tau$, for $\sigma \in \hat{M}_d$, $\tau \in \hat{A}$ and $P_1 = MAN_1$, $P_2 = MAN_2$ associate parabolics [11, p. 473].

III. **Results of Mackey and Anh.** Let X be a locally compact Hausdorff space, μ a finite Borel measure on X, R an equivalence relation on X, $Y = X/R$, and $r\colon X \longrightarrow Y$ the canonical projection. Then R is said to be a measurable equivalence relation if there exists a countable collection $\{E_i\}$ of subsets of Y such that $r^{-1}(E_i)$ is measurable and for every $y \in Y$, $\{y\} = \bigcap \{E_i\colon y \in E_i\}$. Now let G_1, G_2 be closed subgroups of the separable locally compact group G. G_1, G_2 are said to be regularly related if there exist measurable sets $E_0, E_1, \cdots$ in G such that each E_i is a union of $G_1 : G_2$ double cosets, E_0 has measure zero, and every double coset outside of E_0 is the intersection of the E_i that contain it. Let $D =$ the collection of $G_1 : G_2$ double cosets in G. Clearly $G_1 : G_2$ are regularly related if and only if the double cosets outside of a certain set of measure zero form the equivalence classes of a measurable equivalence relation. If μ is a finite measure on G equivalent to Haar measure and $r\colon G \longrightarrow D$ is the projection, then the measure ν given by $\nu(E) = \nu(r^{-1}(E))$, whenever E is such that $r^{-1}(E)$ is measurable, is called an admissible measure on D. Any two such are equivalent. In the special case where there exists a subset of G with complement of measure zero which is itself the countable union of $G_1 : G_2$ double cosets, then $G_1 : G_2$ are called discretely related. In this case ν is a discrete measure.

THEOREM (MACKEY'S SUBGROUP THEOREM [12, p. 127]). *Let G be a separable locally compact group, G_1, G_2 be regularly related closed subgroups of G, and $\pi \in \mathrm{Rep}(G_1)$. For each $x \in G$ consider $G_x = G_2 \cap x^{-1}G_1x$. Form $V_x = \mathrm{Ind}_{G_x}^{G_2}(\eta \longrightarrow \pi(x\eta x^{-1}))$, $\eta \in G_x$. Then V_x is determined to within equivalence by the $G_1 : G_2$ double coset d to which x belongs, write it V_d, and*

$$(\mathrm{Ind}_{G_1}^{G}\ \pi)_{G_2} \simeq \int_D^{\oplus} V_d\, d\nu(d)$$

where ν is any admissible measure on $D = G_1\backslash G/G_2$.

THEOREM (MACKEY'S TENSOR PRODUCT THEOREM [12, p. 128]). *Let G, G_1, G_2 be as in Mackey's subgroup theorem, $\pi_1 \in \mathrm{Rep}(G_1)$, and $\pi_2 \in \mathrm{Rep}(G_2)$. For $(x, y) \in G \times G$ denote*

$$G_{x,y} = x^{-1}G_1x \cap y^{-1}G_2y,$$

$$\pi_{x,y}(g) = \pi_1(xgx^{-1}) \otimes \pi_2(ygy^{-1}), \quad \text{and} \quad \pi^{x,y} = \mathrm{Ind}_{G_{x,y}}^{G}\ \pi_{x,y}.$$

Then $\pi^{x,y}$ is determined to within equivalence by the double coset d to which xy^{-1} belongs, write it π^d, and if ν is any admissible measure on $D = G_1\backslash G/G_2$, then

$$\mathrm{Ind}_{G_1}^{G}\ \pi_1 \otimes \mathrm{Ind}_{G_2}^{G}\ \pi_2 \simeq \int_D^{\oplus} \pi^d\, d\nu(d).$$

The following generalization of Mackey's reciprocity theorem [13, p. 212] will play an important role in §5.

THEOREM (ANH [1, p. 299]). *Let G by a type* I *separable locally compact group, $H \subseteq G$ a closed type* I *subgroup, μ_G, μ_H finite measures in the Plancherel measure classes of G, H respectively, $\omega(\pi, \nu)$ and $n(\pi, \nu)$ be $\mu_G \times \mu_H$ measurable functions where $n(\pi, \nu)$ is a countable cardinal for every π, ν. Then the following are equivalent*:

(i) *For μ_H-almost all ν,*

$$\mathrm{Ind}_H^G\ \nu \simeq \int_{\hat{G}}^{\oplus} n(\pi, \nu)\pi\omega(\pi,\nu)\, d\mu_G(\pi).$$

(ii) *For μ_G-almost all π,*

$$(\pi)_H \simeq \int_{\hat{H}}^{\oplus} n(\pi, \nu)\nu\omega(\pi, \nu)\, d\mu_H(\nu).$$

IV. **Real-rank one Lie groups and algebras.** Let G be a real-rank one connected semisimple Lie group with finite center, Lie algebra $\mathfrak{g}$, and Iwasawa decomposition KAN. Then $\dim A = 1$ and, if α denotes a simple (restricted) root of $\mathfrak{g}$, α may be chosen so that all roots are of the form $j\alpha$, $j = \pm 1, \pm 2$. Let $V = \theta N$, $M =$ the centralizer of A in K, $M' =$ the normalizer of A in K, $P = MAN$, and W the finite (Weyl) group M'/M. W has order two and if $m' \in M' - M$, we have the Bruhat decomposition $G = MAN \cup MANm'MAN$ (and so there are only two $MAN:MAN$ double cosets in G with only one of positive measure and $MANV$ has a complement of Haar measure zero in G). If $\mathfrak{a}$, $\mathfrak{m}$ denote the Lie algebras of A, M and $\mathfrak{g}_{j\alpha} = \{X \in \mathfrak{g}\colon\ [H, X] = j\alpha(H)X$, all $H \in \mathfrak{a}\}$, then

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}.$$

If $\mathfrak{g}$ is simple, then $\mathfrak{g}$ is $\mathrm{so}(n, 1)$, $\mathrm{su}(n, 1)$, $\mathrm{sp}(n, 1)$ or $\mathfrak{f}_{4,9}$ (i.e., G is locally isomorphic to $\mathrm{SO}_e(n, 1)$, $\mathrm{SU}(n, 1)$, $\mathrm{SP}(n, 1)$, or $\mathrm{F}_{4,9}$). If $\mathfrak{g}$ is semisimple, then

$$\mathfrak{g} = \sum_{i=1}^{r} \mathfrak{g}_i = \sum_{1}^{r} (\mathfrak{k}_i + \mathfrak{a}_i + \mathfrak{n}_i)$$

where each $\mathfrak{g}_i$ is a simple ideal [5, p. 122] (and so each $\mathfrak{g}_i$ centralizes $\mathfrak{g}_j$, $i \neq j$) and $\mathfrak{k}_i + \mathfrak{a}_i + \mathfrak{n}_i$ denotes an Iwasawa decomposition for $\mathfrak{g}_i$. Since $\mathfrak{g}$ has real-rank one, all but exactly one of the $\mathfrak{g}_i$, say $\mathfrak{g}_r$, must be compact and so $\mathfrak{g} = (\mathfrak{k}_1 + \mathfrak{k}_2 + \cdots + \mathfrak{k}_r) + \mathfrak{a}_r + \mathfrak{n}_r$ will be an Iwasawa decomposition for $\mathfrak{g}$. Let

$$\widetilde{K} = (\exp \mathfrak{k}_1)(\exp \mathfrak{k}_2) \cdots (\exp \mathfrak{k}_{r-1}).$$

Then

$$G = (\exp \mathfrak{k}_1)(\exp \mathfrak{k}_2) \cdots (\exp \mathfrak{k}_r) A_r N_r = \widetilde{K} K_r A_r N_r,$$

$$K = \widetilde{K} K_r, \quad A = A_r, \quad N = N_r, \quad M = \widetilde{K} M_r.$$

Note that since $\mathrm{Ad}(\widetilde{K})$ fixes $\mathfrak{n} = LA(N)$ (or $\mathfrak{v} = \theta\,\mathfrak{n}$), in determining the orbits of points in $\mathfrak{n}$ (or $\mathfrak{v}$) under the action of MA, we may assume that G is simple (of course the stability groups will in general be larger). Let us examine the simple real-rank one groups more closely.

A. *The classical cases.* We shall look more closely at $\mathrm{SO}_e(n, 1)$, $\mathrm{SU}(n, 1)$, $\mathrm{SP}(n, 1)$ for $n \geq 2$ and $\mathrm{Spin}(n, 1)$ for $n \geq 3$ (recall that $\mathrm{SU}(1, 1)$ is locally isomorphic to $\mathrm{SO}_e(2, 1)$, $\mathrm{SP}(1, 1)$ is locally isomorphic to $\mathrm{SO}_e(4, 1)$ [5, p. 351], and $\mathrm{SO}_e(1, 1) \approx \mathbf{R}^x$ is not semisimple).

Let $n \geq 2$ and $\mathbf{K}$ be $\mathbf{R}, \mathbf{C}$ or H (the quaternions). Let G be the group of all automorphisms of $\mathbf{K}^{n+1}$ which preserve the hermitian quadratic form $|x_1|^2 + \cdots + |x_n|^2 - |x_{n+1}|^2$ with the additional property that if $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, we consider only automorphisms of determinant 1. Then G is $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$ or $\mathrm{SP}(n, 1)$ according to whether $\mathbf{K}$ is $\mathbf{R}, \mathbf{C}$ or H [8, p. 555]. $\mathrm{SU}(n, 1)$ and $\mathrm{SP}(n, 1)$ are connected [5, p. 346] and we denote by $\mathrm{SO}_e(n, 1)$ the identity component of $\mathrm{SO}(n, 1)$. Then we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Lie algebra of G where:

$$\mathfrak{k} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix},$$ X_1 is an $n \times n$ skew hermitian with entries in $\mathbf{K}$, X_2 is a skew member of $\mathbf{K}$, $X_2 + \operatorname{tr} X_1 = 0$ if $\mathbf{K} = \mathbf{C}$ ($X_2 = 0$ for $\mathbf{K} = \mathbf{R}$),

$$\mathfrak{p} = \begin{pmatrix} 0 & Y \\ \bar{Y}^t & 0 \end{pmatrix}, \quad Y \text{ is a column vector in } \mathbf{K}^n.$$

In each case the Cartan involution, θ, is negative conjugate transpose. We also have:

$$\mathfrak{a} = \mathbf{R} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix},$$

$$\mathfrak{g}_\alpha = \begin{pmatrix} 0 & \bar{X}^t & 0 \\ -X & 0 & X \\ 0 & \bar{X}^t & 0 \end{pmatrix}, \quad X \text{ a column vector in } \mathbf{K}^{n-1},$$

$$\mathfrak{g}_{2\alpha} = \begin{pmatrix} Y & 0 & -Y \\ 0 & 0 & 0 \\ Y & 0 & -Y \end{pmatrix}, \quad \mathfrak{g}_{2\alpha} = 0 \text{ if } \mathbf{K} = \mathbf{R};\ Y \in \mathbf{K} \text{ with } \bar{Y} = -Y \text{ if } \mathbf{K} = \mathbf{C} \text{ or } H,$$

$$\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}, \qquad \mathfrak{v} = \theta\mathfrak{n},$$

$$V = \begin{pmatrix} 1 + Y - \tfrac{1}{2}|X|^2 & \bar{X}^t & Y - \tfrac{1}{2}|X|^2 \\ -X & I & -X \\ -Y + \tfrac{1}{2}|X|^2 & -\bar{X}^t & 1 - Y + \tfrac{1}{2}|X|^2 \end{pmatrix},$$

$$N = \begin{pmatrix} 1 + Y - \tfrac{1}{2}|X|^2 & \bar{X}^t & -Y + \tfrac{1}{2}|X|^2 \\ -X & I & X \\ Y - \tfrac{1}{2}|X|^2 & \bar{X}^t & 1 - Y + \tfrac{1}{2}|X|^2 \end{pmatrix},$$

$$K = \begin{cases} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \approx \mathrm{SO}(n), \quad k \in \mathrm{SO}(n) \text{ for } \mathbf{K} = \mathbf{R}, \\ \begin{pmatrix} u & 0 \\ 0 & c \end{pmatrix} \approx S(U(n) \times U(1)) \approx U(n), \quad u \in U(n),\ |c| = 1, \\ \qquad c \det u = 1 \text{ for } \mathbf{K} = \mathbf{C}, \\ \begin{pmatrix} \omega & 0 \\ 0 & u \end{pmatrix} \approx \mathrm{Sp}(n) \times \mathrm{Sp}(1), \quad \omega \in \mathrm{Sp}(n),\ u \in \mathrm{Sp}(1) \\ \qquad \text{(unit quaternions) for } \mathbf{K} = H, \end{cases}$$

$$\mathfrak{m} = \begin{pmatrix} X_2 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X_2 \end{pmatrix}, \quad X_2 \text{ skew in } \mathbf{K}\ (= 0 \text{ if } \mathbf{K} = \mathbf{R}); \ X \text{ an } (n-1) \times (n-1) \text{ skew hermitian with entries in } \mathbf{K},\ 2X_2 + \operatorname{tr} X = 0 \text{ if } \mathbf{K} = \mathbf{C},$$

$$M=\begin{cases}\begin{pmatrix}1 & 0 & 0\\ 0 & u & 0\\ 0 & 0 & 1\end{pmatrix}, & u\in SO(n-1) \text{ for } \mathbf{K}=\mathbf{R},\\ \begin{pmatrix}c & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & c\end{pmatrix}, & |c|=1,\ \lambda\in U(n-1),\ c^2\det\lambda=1 \text{ for } \mathbf{K}=\mathbf{C},\\ \begin{pmatrix}u & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & u\end{pmatrix}, & u\in Sp(1),\ \omega\in Sp(n-1) \text{ for } \mathbf{K}=H.\end{cases}$$

To investigate the action of MA on V, we deal with an algebra conjugate to $\mathfrak{g}$. Let

$$s_0=\begin{pmatrix}2^{-1/2} & 0 & 2^{-1/2}\\ 0 & I & 0\\ -2^{-1/2} & 0 & 2^{-1/2}\end{pmatrix}$$

and $\mathfrak{g}=s_0\mathfrak{g}s_0^{-1}$. Then A becomes

$$\begin{pmatrix}e^t & 0 & 0\\ 0 & I & 0\\ 0 & 0 & e^{-t}\end{pmatrix},$$

M remains the same,

$$N=\begin{pmatrix}1 & \bar{X}^t & \tfrac12(Y+|X|^2)\\ 0 & I & X\\ 0 & 0 & 1\end{pmatrix},$$

and

$$V=\exp(X+Y)\equiv(X,Y)=\begin{pmatrix}1 & 0 & 0\\ X & I & 0\\ \tfrac12(Y+|X|^2) & \bar{X}^t & 1\end{pmatrix}.$$

The action of A on V is:

$$\begin{pmatrix}e^t & 0 & 0\\ 0 & I & 0\\ 0 & 0 & e^{-t}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ X & I & 0\\ \tfrac12(Y+|X|^2) & \bar{X}^t & 1\end{pmatrix}\begin{pmatrix}e^{-t} & 0 & 0\\ 0 & I & 0\\ 0 & 0 & e^t\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ e^{-t}X & I & 0\\ e^{-2t}\tfrac12(Y+|X|^2) & e^{-t}\bar{X}^t & 1\end{pmatrix}$$

i.e., $a \cdot v = e^t \cdot (X, Y) = (e^{-t}X, e^{-2t}Y)$ while the action of M on V is:

$$\begin{pmatrix} u & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ X & I & 0 \\ \tfrac{1}{2}(Y + |X|^2) & \bar{X}^t & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 & 0 \\ 0 & \omega^{-1} & 0 \\ 0 & 0 & u^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \omega X u^{-1} & I & 0 \\ u\tfrac{1}{2}(Y + |X|^2)u^{-1} & u\bar{X}^t\omega^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \omega X u^{-1} & I & 0 \\ \tfrac{1}{2}(uYu^{-1} + |X|^2) & u\bar{X}^t\omega^{-1} & 1 \end{pmatrix}$$

i.e., $m \cdot v = \operatorname{diag}(u, \omega, u) \cdot (X, Y) = (\omega X u^{-1}, uYu^{-1})$. In each case the action of M is by rotation while the action of A is by dilation. For $\mathbf{K} = \mathbf{R}$, MA acts on $V^x = V - \{0\}$ in one orbit for $n \geqslant 3$ and two orbits for $n = 2$. For $\mathbf{K} = \mathbf{C}$, MA acts on $\exp \mathfrak{g}_{-\alpha}$ by rotation and dilation by e^{-t} while only A acts on $\exp \mathfrak{g}_{-2\alpha}$ (M fixes $\exp \mathfrak{g}_{-2\alpha}$) by dilation by e^{-2t}. So MA acts transitively on $\exp \mathfrak{g}^x_{-\alpha}$ while MA acts on $\exp \mathfrak{g}^x_{-2\alpha}$ in two orbits. If we take $X_{-\alpha} \in \mathfrak{g}^x_{-\alpha} = \mathfrak{g}_{-\alpha} - \{0\}$, $Y_{-2\alpha} \in \mathfrak{g}^x_{-2\alpha}$, then up to a set of Haar measure zero in V, $\{\exp(tX_{-\alpha} \pm Y_{-2\alpha})\colon t > 0\}$ will serve as a cross-section for the action of MA on $V = \exp(\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha})$. For $\mathbf{K} = H$, $M = \mathrm{Sp}(1) \times \mathrm{Sp}(n-1)$ where both $\mathrm{Sp}(1)$ and $\mathrm{Sp}(n-1)$ act on $\exp \mathfrak{g}_{-\alpha}$ while only $\mathrm{Sp}(1)$ acts on $\exp \mathfrak{g}_{-2\alpha}$. One may easily verify that the above action of $\mathrm{Sp}(1)$ on $\exp \mathfrak{g}_{-2\alpha}$ is the same as that of $\mathrm{SO}(3)$ on $\mathbf{R}^3$ [2, p. 38]. In this case MA will act transitively on both $\exp \mathfrak{g}^x_{-\alpha}$ and $\exp \mathfrak{g}^x_{-2\alpha}$, in fact if $M_Y = \{m \in M\colon m \cdot \exp Y = \exp Y,\ Y \in \mathfrak{g}^x_{-2\alpha}\}$, then $M_Y A$ also acts transitively on $\exp \mathfrak{g}^x_{-\alpha}$. So if we take $X_{-\alpha} \in \mathfrak{g}^x_{-\alpha}$, $Y_{-2\alpha} \in \mathfrak{g}^x_{-2\alpha}$, then up to a set of Haar measure zero in V, $\{\exp(tX_{-\alpha} + Y_{-2\alpha})\colon t > 0\}$ will be a cross-section for the action of MA on V.

For $n \geqslant 3$, $\mathrm{Spin}(n, 1)$ denotes the simply-connected double covering group of $\mathrm{SO}_e(n, 1)$. (Recall that the universal covering of $\mathrm{SL}(2, \mathbf{R})$ or $\mathrm{SU}(n, 1)$ does not have a finite center and that $\mathrm{Sp}(n, 1)$, as well as $\mathrm{F}_{4,9}$, is already simply connected.) If $\widetilde{G} = \mathrm{Spin}(n, 1) = \widetilde{K}\widetilde{A}\widetilde{N}$ and $G = \mathrm{SO}_e(n, 1) = KAN$, then we have $\widetilde{K} = \mathrm{Spin}(n)$, $\widetilde{K}/Z_2 \approx K$, $\widetilde{A} = A$, $\widetilde{N} = N$, $\widetilde{M} = \mathrm{Spin}(n-1)$,. and $\widetilde{M}/Z_2 \approx M$. Clearly $\widetilde{M}A$ acts transitively on V^x.

B. *The exceptional case*: $\mathrm{F}_{4,9}$. There is only one nonclassical simple Lie group of real-rank one. It is a real form of F_4 [5, p. 354]. Denote this group by $\mathrm{F}_{4,9}$ and its Lie algebra by $\mathfrak{f}_{4,9}$. Then $\dim \mathfrak{f}_{4,9} = 52$, $\dim \mathfrak{g}_{2\alpha} = 7$, $\dim \mathfrak{g}_\alpha = 8$, $K = \mathrm{Spin}(9)$, and $M = \mathrm{Spin}(7)$ (see [6]).

C. The only remaining simple groups of real-rank one are quotients of the universal covering groups of the above groups by discrete central subgroups Z

($\subseteq M$). Note that if $G = KAN$ is such a universal covering group, Z is a discrete central subgroup of G, and $G' = G/Z = K'A'N'$, then

$$K' = K/Z, A' = A, N' = N, \quad \text{and} \quad M' = M/Z.$$

In particular for $m' = mZ \in M$, $X \in \mathfrak{n}$ (or $\mathfrak{v}$), $\text{Ad}(m')X = \text{Ad}(mZ)X = \text{Ad}(m)X$ and so in computing the orbits of points $\mathfrak{n}$ (or $\mathfrak{v}$) under the action of $M'A'$, we may assume that G' or G is one of the groups described in A or B, i.e., $\text{SO}_e(n, 1), \text{SU}(n, 1), \text{Sp}(n, 1)$, or $\text{F}_{4,9}$.

Let us remark at this time that since MA can be regarded as a direct product, the irreducible representations $L \in (MA)^\wedge$ are all of the form $L = \sigma \times \tau$ with $\sigma \in \hat{M}, \tau \in \hat{A}$ and so $(MA)^\wedge = \hat{M} \times \hat{A}$ with Plancherel measure on $(MA)^\wedge$ being the product of the Plancherel measures on $\hat{M}$ and $\hat{A}$ [7]. We shall put $MA = C$ where convenient and shall denote by μ_C (resp. μ_G) Plancherel measure on $\hat{C}$ (resp. $\hat{G}$). We now thank Anthony Knapp for supplying the details of the following argument which is due to Kostant and lies behind some of this work in [9].

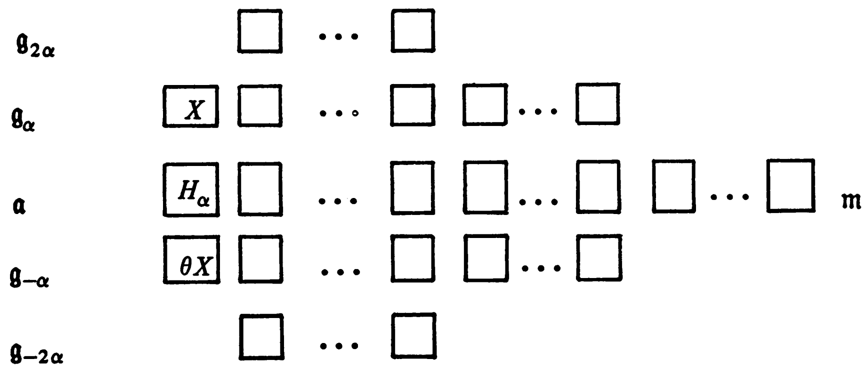
LEMMA. *Let* $\mathfrak{g}$ *be a real-rank one semisimple Lie algebra and* $k = \pm 1, \pm 2$. *Then*

(i) *if* $\dim \mathfrak{g}_{k\alpha} = 1$, *then* MA *will act on* $\mathfrak{g}_{k\alpha} - \{0\} = \mathfrak{g}_{k\alpha}^{x}$ *in two orbits while*

(ii) *if* $\dim \mathfrak{g}_{k\alpha} > 1$, *then* MA *will act on* $\mathfrak{g}_{k\alpha}^{x}$ *in one orbit.*

PROOF. As stated earlier in this section, we may assume $\mathfrak{g}$ is simple. Note also that the above lemma follows immediately from previous calculations when $\mathfrak{g}$ is one of the classical real-rank one algebras. The following general argument shows the result to be true for $\mathfrak{f}_{4,9}$ as well. As noted above, we may assume $\dim \mathfrak{g}_{k\alpha} > 1$ for $k = \pm 1, \pm 2$.

Let $k = 1$, $X \in \mathfrak{g}_\alpha$, and $H_\alpha \in \mathfrak{a}$ be determined by $\alpha(H) = B_\theta(H, H_\alpha)$, $H \in \mathfrak{a}$. Then $\theta X \in \mathfrak{g}_{-\alpha}$, $[X, \theta X] = B(X, \theta X)H_\alpha \in \mathfrak{a}$, and $\{X, \theta X, H_\alpha\}$ spans a subalgebra of $\mathfrak{g}$ isomorphic to $\text{sl}(2, \mathbf{R})$. Left multiplication by this subalgebra gives a representation of $\text{sl}(2, \mathbf{R})$ on $\mathfrak{g}$ and $\{X, \theta X, H_\alpha\}$ spans an invariant subspace. Any finite-dimensional representation of $\text{sl}(2, \mathbf{R})$ splits into a direct sum of irreducibles and we may assume that these irreducibles are orthogonal with respect to the natural inner product on $\mathfrak{g}$. One of these irreducibles is $\{X, \theta X, H_\alpha\}$ with weights (i.e., the eigenvalues of $\text{ad}\, H_\alpha$) $\alpha, 0, -\alpha$. Matching these weights with the weights of all the irreducible abstract representations of $\text{sl}(2, \mathbf{R})$, we see that the weights of an n-dimensional representation would have to be $\tfrac{1}{2}(n-1)\alpha, \tfrac{1}{2}(n-3)\alpha, \cdots, -\tfrac{1}{2}(n-1)\alpha$. The only possible weights in $\mathfrak{g}$ are $2\alpha, \alpha, 0, -\alpha, -2\alpha$ and so $n = 1, 3$ or 5. We now have the following diagram:



where each vertical column represents an invariant subspace and each box represents a one-dimensional subspace. Note that $\operatorname{ad} X$ of each box in the $\mathfrak{a} + \mathfrak{m}$ row is contained in the corresponding box in the $\mathfrak{g}_\alpha$ row. Let $X = X_1, \cdots, X_p$ be a basis for the $\mathfrak{g}_\alpha$ row and choose basis elements $H_1, \cdots, H_p, H_{p+1}, \cdots, H_q$ in the $\mathfrak{a} + \mathfrak{m}$ row such that $[X, H_i] = X_i$, $i = 1, \cdots, p$ (e.g., $H_1 = -H_\alpha/\alpha(H_\alpha)$). Any vector $Y \in \mathfrak{g}_\alpha$ is a linear combination of the X_i's, say $Y = \Sigma_{i=1}^p c_i X_i$. Let $W = \Sigma_{i=1}^p c_i H_i \in \mathfrak{a} + \mathfrak{m}$. Then

$$[X, W] = \sum_{i=1}^{p} c_i [X, H_i] = \sum_{i=1}^{p} c_i X_i = Y$$

and we have shown that $[X, \mathfrak{a} + \mathfrak{m}] = \mathfrak{g}_\alpha$ for $X \in \mathfrak{g}_\alpha - \{0\}$.

Now we consider the map $f\colon MA \longrightarrow \mathfrak{g}_\alpha$ via $ma \mapsto \operatorname{Ad}(ma)X$. Since $df\colon \mathfrak{a} + \mathfrak{m} \longrightarrow \mathfrak{g}_\alpha$ is given by $m + a \mapsto \operatorname{ad}(m + a)X = [m + a, X]$, the above argument shows that df is onto and so it follows from the inverse function theorem that the orbit $\operatorname{Ad}(MA)X$ is open in $\mathfrak{g}_\alpha - \{0\}$. Since M is compact and the action of A is given by

$$a \cdot X = \operatorname{Ad}(a)X = e^{\operatorname{ad}(H)}X = e^{\alpha(H)}X, \qquad a = \exp H,\ H \in \mathfrak{a},$$

it follows that this orbit is also closed in $\mathfrak{g}_\alpha - \{0\}$. Since $\dim \mathfrak{g}_\alpha > 1$, $\operatorname{Ad}(MA)$ acts in $\mathfrak{g}_\alpha - \{0\}$ in one orbit.

The proofs for $k = -1, \pm 2$ are similar and the lemma has been proven.

Note. Let $c \in MA$, $X \in \mathfrak{g}_{k\alpha}$. Since $c \cdot \exp X = c(\exp X)c^{-1} = \exp \operatorname{Ad}(c)X = \exp(c \cdot X)$, there is a canonical 1-1 correspondence between orbits in $\mathfrak{g}_{k\alpha}/MA$ and orbits in $(\exp \mathfrak{g}_{k\alpha})/MA$. Thus MA will act on $\exp(\mathfrak{g}_{k\alpha}^x)$ in one orbit for $\dim \mathfrak{g}_{k\alpha} > 1$ and two orbits for $\dim \mathfrak{g}_{k\alpha} = 1$.

V. **The tensor product of principal series representations.** Let G be a connected semisimple Lie group with finite center, $P_1 = MAN$ be a cuspidal parabolic subgroup of G, and $P_2 = MA\bar{N}$ where $\bar{N} = \theta N$. Recall that $\hat{M}_d$ denotes the square-integrable irreducible representations of M and for $\tau \in \hat{A}$,

$\sigma \in \hat{M}_d$, $\sigma \times \tau$ is a representation of MAN via $(\sigma \times \tau)(man) = \sigma(m)\tau(a)$. For $\sigma \in \hat{M}_d$, $\tau \in \hat{A}$ let $\pi(\sigma, \tau) = \mathrm{Ind}_{P_1}^G\ \sigma \times \tau$.

THEOREM 1.

$$\pi(\sigma_1, \tau_1) \otimes \pi(\sigma_2, \tau_2) \simeq \mathrm{Ind}_{MA}^G\ (\sigma_1 \otimes \sigma_2)(\tau_1 \tau_2)$$

where $\sigma_1 \in \hat{M}_d$, $\tau_i \in \hat{A}$, $i = 1, 2$.

PROOF. Since P_1, P_2 are associate parabolic subgroups, we have (see [11, p. 473]) $\mathrm{Ind}_{P_1}^G\ \sigma_2 \times \tau_2 \simeq \mathrm{Ind}_{P_2}^G\ \sigma_2 \times \tau_2$ and we may apply Mackey's tensor product theorem with $G_i = P_i$, $i = 1, 2$. P_1, P_2 are discretely related, there is exactly one double coset of positive measure in G, and if we take e as a representative for this double coset, we have $P_1 \cap P_2 = MA$. Thus

$$\begin{aligned} \pi(\sigma_1, \tau_1) \otimes \pi(\sigma_2, \tau_2) &\simeq \mathrm{Ind}_{P_1}^G\ \sigma_1 \times \tau_1 \otimes \mathrm{Ind}_{P_2}^G\ \sigma_2 \times \tau_2 \\ &\simeq \mathrm{Ind}_{MA}^G\ (\sigma_1 \otimes \sigma_2)(\tau_1 \tau_2). \end{aligned}$$

Thus the problem of decomposing the tensor product of two continuous series representations arising from the same cuspidal parabolic subgroup can be solved once one knows how to decompose $\sigma_1 \otimes \sigma_2$ for $\sigma_1, \sigma_2 \in \hat{M}_d$ and how to decompose $\mathrm{Ind}_{MA}^G\ L$ for all $L \in (MA)^\wedge$. However, there is no reason, a priori, to expect that either of these last two problems is any easier than the one we started with. In what follows, we show to what extent these problems can be dealt with when P is a minimal parabolic subgroup and in particular when G has real-rank one.

THEOREM 2. *Let* $P = MAN$ *be a minimal parabolic subgroup. Then for* $\sigma \in \hat{M}$, $\tau_1, \tau_2 \in \hat{A}$ *we have*

$$\mathrm{Ind}_{MA}^G\ \sigma \times \tau_1 \simeq \mathrm{Ind}_{MA}^G\ \sigma \times \tau_2.$$

PROOF. Let G have real-rank $l \geqslant 1$ and let $\alpha_1, \cdots, \alpha_l$ denote the simple (restricted) roots of $\mathfrak{g}$. Since the simple roots form a basis for $\mathfrak{a}^*$ and $\hat{A} \approx \mathfrak{a}^*$, any $\tau' \in \hat{A}$ is of the form $\tau'(a) = e^{i\beta(H)}$ where $\beta = \Sigma_{i=1}^l m_i \alpha_i$, $m_i \in \mathbf{R}$, and $a = \exp H$, $H \in \mathfrak{a}$. Let $\tau^j(a) = e^{i m_j \alpha_j(H)}$ for $j = 1, \cdots, l$ and $\tau^0 = 1$ where 1 is the 1-dimensional identity representation of A (i.e., the representation corresponding to $0 \in \mathfrak{a}^*$). Then $\tau^i \in \hat{A}$, $i = 0, 1, \cdots, l$, and $\tau' = \Sigma_{i=0}^l \tau^i$. We shall show that

$$\begin{aligned} \mathrm{Ind}_{MA}^G\ \sigma \times \tau' &\simeq \mathrm{Ind}_{MA}^G\ \sigma \times \prod_{i=0}^{l-1} \tau^i \simeq \cdots \\ &\simeq \mathrm{Ind}_{MA}^G\ \sigma \times \tau^1 \simeq \mathrm{Ind}_{MA}^G\ \sigma \times 1. \end{aligned}$$

It clearly suffices to show

$$(*) \qquad \operatorname{Ind}_{MA}^{G} \sigma \times \tau \simeq \operatorname{Ind}_{MA}^{G} \sigma \times \prod_{i=0}^{j-1} \tau^i$$

where $\tau = \Pi_{i=0}^{j} \tau^i$ and $j = 1, \cdots, l$. Fix $\alpha = \alpha_j$ and let

$$\mathfrak{n}_\alpha = \sum_{k\geq 1} \mathfrak{g}_{k\alpha}, \qquad \mathfrak{v}_\alpha = \theta\mathfrak{n}_\alpha = \sum_{k\geq 1} \mathfrak{g}_{-k\alpha}, \qquad \tau_- = \prod_{i=0}^{j-1} \tau^i.$$

Then, as in [10, p. 399], the subalgebra $\mathfrak{g}^\alpha$ of $\mathfrak{g}$ generated by $\mathfrak{n}_\alpha$ and $\mathfrak{v}_\alpha$ is of the form $\mathfrak{g}^\alpha = \mathfrak{a}_\alpha \oplus \mathfrak{m}_\alpha \oplus \mathfrak{n}_\alpha \oplus \mathfrak{v}_\alpha$ where $\mathfrak{m}_\alpha \subseteq \mathfrak{m} = LA(M)$ and $\mathfrak{a}_\alpha = \mathbf{R} \cdot H_\alpha$. So $\mathfrak{g}^\alpha$ is a real-rank one semisimple subalgebra of $\mathfrak{g}$. Let $\mathfrak{q} = \mathfrak{g}_\alpha$ or $\mathfrak{q} = \mathfrak{g}_{2\alpha}$ according to whether $\mathfrak{g}_{2\alpha} = \{0\}$ or not. If $Q = \exp \mathfrak{q}$, then Q is an abelian subgroup of N such that MA leaves Q invariant. In fact, from the lemma of §4, we know that $\exp(\mathfrak{m}_\alpha \oplus \mathfrak{a}_\alpha)$ will act on $Q - \{e\}$ in two orbits or one orbit depending upon whether $\dim Q = 1$ or not. If $\sigma \in \hat{M} = \hat{M}_d$ (M is compact), then (*) will follow by induction in stages [12, p. 113] once we show

$$(**) \qquad \pi_\tau = \operatorname{Ind}_{MA}^{MAQ} \sigma \times \tau \simeq \operatorname{Ind}_{MA}^{MAQ} \sigma \times \tau_- = \pi_-.$$

We first say something about the action of MA on $\mathfrak{q}^*$ (via the coadjoint representation) and then define a function D: $\mathfrak{q}^* \longrightarrow \Pi$ which is to satisfy the important identity

$$D((ma)^{-1} \cdot \phi) = D(\phi)\tau^j(a) \quad \text{for all} \quad \phi \in \mathfrak{q}^* - \{0\},\ ma \in MA.$$

In the following, we let $k\alpha$ be 2α or α according to whether $\mathfrak{q} = \mathfrak{g}_{2\alpha}$ or not. Let $(\cdot, \cdot)$ be an $\operatorname{Ad}(M)$-invariant inner product on $\mathfrak{q}$ and note that $(a \cdot X, Y) = (X, a \cdot Y) = e^{k\alpha(H)}(X, Y)$ where $a = \exp H$, $H \in \mathfrak{a}$, and $X, Y \in \mathfrak{q}$. Define a map U: $\mathfrak{q}^* \longrightarrow \mathfrak{q}$ via $\phi \mapsto X_\phi$ where $\phi(X) = (X, X_\phi)$ for $X \in \mathfrak{q}$. Then U is a vector space isomorphism (and so Borel). By definition we have $(ma \cdot \phi)X = \phi((ma)^{-1} \cdot X)$. So for all $X \in \mathfrak{q}$ we have

$$\begin{aligned}(X, X_{ma\cdot\phi}) &= (ma \cdot \phi)(X) = \phi((m^{-1}a^{-1}) \cdot X)\\ &= (m^{-1}a^{-1} \cdot X, X_\phi) = (a^{-1} \cdot X, m \cdot X_\phi) = (X, ma^{-1} \cdot X_\phi),\end{aligned}$$

i.e., $X_{ma\circ\phi} = ma^{-1} \cdot X_\phi$. Note that if we define $(\phi_1, \phi_2)_* = (X_{\phi_1}, X_\phi)$ for $\phi_1, \phi_2 \in \mathfrak{q}^*$, then $(\cdot, \cdot)_*$ is an $\operatorname{Ad}^*(M)$-invariant inner product on $\mathfrak{q}^*$. Let Φ_0 be fixed in $\mathfrak{q}^* - \{0\}$.

If $\dim \mathfrak{q} > 1$, then we know that MA acts transitively on $\mathfrak{q} - \{0\}$. Let $\phi \in \mathfrak{q}^* - \{0\}$ and choose $ma^{-1} \in MA \ni ma^{-1} \cdot X_{\phi_0} = X_\phi$. Then

$$\begin{aligned}(ma \cdot \phi_0)(X) &= \phi_0(m^{-1}a^{-1} \cdot X) = (m^{-1}a^{-1} \cdot X, X_{\phi_0})\\ &= (X, ma^{-1} \cdot X_{\phi_0}) = (X, X_\phi) = \phi(X)\end{aligned}$$

and so $ma \circ \phi_0 = \phi$. Thus MA also acts transitively on $\mathfrak{q}^* - \{0\}$. So for $\dim \mathfrak{q} > 1$, define D: $\mathfrak{q}^* \longrightarrow \Pi$ by $D(0) = 1$, $D((ma)^{-1} \cdot \phi_0) = \tau^j(a)$. Note that D is defined on all of $\mathfrak{q}^*$ and is a Borel function. Note also that

(1) D is well-defined.

Suppose $(ma)^{-1} \cdot \phi_0 = \phi_0$ with $a = \exp H$, $H \in \mathfrak{a}$. Since $(a \cdot \phi_0)(X) = (a^{-1} \cdot X, X_{\phi_0}) = e^{-k\alpha(H)}(X, X_{\phi_0}) = e^{-k\alpha(H)}\phi_0(X)$, we have $a \cdot \phi_0 = e^{-k\alpha(H)}\phi_0$. Let $\|\phi\| = (\phi, \phi)_*^{1/2}$. Then $\|m^{-1}a^{-1} \cdot \phi_0\| = \|a^{-1} \cdot \phi_0\| = e^{k\alpha(H)}\|\phi_0\| = \|\phi_0\|$ implies that $k\alpha(H) = 0$. Hence $\tau^j(a) = e^{im_jk\alpha(H)} = 1$ and D is well-defined.

(2) $D((ma)^{-1} \cdot \phi) = D(\phi)\tau^j(a)$ for $\phi \in \mathfrak{g}^*_{k\alpha} - \{0\}$, $ma \in MA$. Let $\phi \in \mathfrak{q}^* - \{0\}$ and choose $m_1a_1 \in MA$ $(m_1a_1)^{-1} \cdot \phi_0 = \phi$. Then

$$D((ma)^{-1} \cdot \phi) = D((mm_1aa_1)^{-1} \cdot \phi_0) = \tau^j(aa_1)$$

$$= \tau^j(a)\tau^j(a_1) = D(\phi)\tau^j(a).$$

Now suppose $\dim \mathfrak{q} = 1$. We know $\exp(\mathfrak{a}_\alpha \oplus \mathfrak{m}_\alpha)$ acts on $\mathfrak{q} - \{0\}$ in two orbits (although it is conceivable that MA acts in one orbit), in fact, we know that $a \cdot X = e^{k\alpha(H)}X$ for $X \in \mathfrak{q} - \{0\}$, $a = \exp H$, $H \in \mathfrak{a}$ and that $a \cdot \phi = e^{-k\alpha(H)}\phi$ for $\phi \in \mathfrak{q}^* - \{0\}$. So A acts on $\mathfrak{q}^* - \{0\}$ in two orbits. Define D: $\mathfrak{q}^* \longrightarrow \Pi$ by $D(0) = 1$, $D((ma)^{-1} \cdot (\pm\phi_0)) = \tau^j(a)$. Then D is Borel, defined on all of $\mathfrak{q}^*$, and

(1) D is well-defined:

Suppose $(ma)^{-1} \cdot (\pm\phi_0) = \pm\phi_0$. Then, as above, $k\alpha(H) = 0$ and $\tau^j(a) = 1$.

(2) $D((ma)^{-1} \cdot \phi) = D(\phi)\tau^j(a)$ for $\phi \in \mathfrak{q}^* - \{0\}$, $ma \in MA$. Let $\phi \in \mathfrak{q}^* - \{0\}$ and choose $a_1 \in A \ni a_1^{-1} \cdot (\pm\phi_0) = \phi$. Then

$$D((ma)^{-1} \cdot \phi) = D((maa_1)^{-1} \cdot (\pm\phi_0)) = \tau^j(aa_1) = D(\phi)\tau^j(a).$$

Let dX denote Lebesgue measure on $\mathfrak{q}$ and choose Haar measure dq on Q such that $\int_Q f(q)dq = \int_\mathfrak{q} f(\exp X)dX$ for $f \in L^1(Q, dq)$. If we let ψ: $L^2(Q, dq) \longrightarrow L^2(\mathfrak{q}, dX)$ via $(\psi f)(X) = f(\exp X)$, then ψ is a unitary operator between $L^2(Q)$ and $L^2(\mathfrak{q})$. For $ma \in MA$, let $\Delta_{k\alpha}(ma)$ denote the modulus of the automorphism $q \mapsto (ma)q(ma)^{-1}$, $q \in Q$. As in [10, p. 392], since $m \mapsto \Delta_{k\alpha}(m)$ defines a continuous homomorphism of the compact group $M/\text{center}(G)$ into the multiplicative group of positive reals, we have that $\Delta_{k\alpha}(m) = 1$, for all $m \in M$. Now let $a = \exp H$, $H \in \mathfrak{a}$ and $\dim \mathfrak{q} = c_{k\alpha}$. Then

$$\Delta_{k\alpha}(a) = \det(\mathrm{Ad}(a)) = \det(\exp \mathrm{ad}\, H) = \exp(\mathrm{tr}\, \mathrm{ad}\, H)$$

$$= \exp(c_{k\alpha}k\alpha(H)).$$

So

$$\int_Q f(q)dq = \Delta_{k\alpha}(a) \int_Q f((ma)q(ma)^{-1})dq.$$

Now

$$\begin{aligned}\int_{\mathfrak{q}} (\Psi f)(\operatorname{Ad}(ma)X)\,dX &= \int_{\mathfrak{q}} f(\exp \operatorname{Ad}(ma)X)\,dX \\ &= \int_Q f((ma)q(ma)^{-1})\,dq \\ &= \Delta_{k\alpha}(a^{-1}) \int_Q f(q)\,dq = \Delta_{k\alpha}(a^{-1}) \int_{\mathfrak{q}} (\Psi f)(X)\,dX.\end{aligned}$$

Thus the modulus of $X \mapsto \operatorname{Ad}(ma)X$ is also $\Delta_{k\alpha}(a) = e^{c_{k\alpha}k\alpha(H)}$.

We now prove (**). If $f \in L^2(Q, H_\sigma)$, then

$$\pi_\tau(man)f(q) = \sigma(m)\tau(a)f(m^{-1}a^{-1}qamn)\Delta_{k\alpha}^{-1/2}(a)$$

and

$$\pi\_(man)f(q) = \sigma(m)\prod_{i=0}^{j-1} \tau^i(a)f(m^{-1}a^{-1}gamn)\Delta_{k\alpha}^{-1/2}(a)$$

where $man \in MAQ$. Now let Ψ: $L^2(Q, H_\sigma) \longrightarrow L^2(\mathfrak{q}, H_\sigma)$ by $(\Psi f)(X) = f(\exp X)$. Then Ψ is a unitary operator and for $h \in L^2(\mathfrak{q}, H_\sigma)$ with $\Psi f = h$ we have

$$\begin{aligned}\tilde{\pi}_\tau(man)h(X) \equiv (\Psi\pi_\tau(man)\psi^{-1}h)(X) &= (\psi\pi_\tau(man)f)(X) \\ &= \pi_\tau(man)f(\exp X) \\ &= \sigma(m)\tau(a)f(m^{-1}a^{-1}(\exp X)am \exp X_n)\Delta_{k\alpha}^{-1/2}(a), \quad n = \exp X_n, \\ &= \sigma(m)\tau(a)f((\exp \operatorname{Ad}(ma)^{-1}X)\exp X_n)\Delta_{k\alpha}^{-1/2} \\ &= \sigma(m)\tau(a)f(\exp(\operatorname{Ad}(ma)^{-1}X + X_n))\Delta_{k\alpha}^{-1/2}(a) \\ &= \sigma(m)\tau(a)h(\operatorname{Ad}(ma)^{-1}X + X_n)\Delta_{k\alpha}^{-1/2}(a).\end{aligned}$$

Similarly

$$\tilde{\pi}\_(man)h(x) = \sigma(m)\prod_{i=0}^{j-1} \tau^i(a)h(\operatorname{Ad}(ma)^{-1}X + X_n)\Delta_{k\alpha}^{-1/2}(a).$$

For $X \in \mathfrak{q}$, $f \in L^2(\mathfrak{q}, H_\sigma)$, and $\phi \in \mathfrak{q}^*$ we have

$$(\mathcal{F}f)(\phi) = \int_{\mathfrak{q}} f(X)e^{i\phi(X)}\,dX,$$

the operator-valued Fourier transform on $\mathfrak{q}$. We now use $\mathcal{F}$ to realize $\tilde{\pi}_\tau$ and $\tilde{\pi}\_$ on $L^2(\mathfrak{q}^*, H_\sigma)$. Let $H \in L^2(\mathfrak{q}^*, H_\sigma)$ with $\mathcal{F}h = H$.

$$
\begin{aligned}
\hat{\pi}_\tau(man)H(\phi) &= (F\tilde{\pi}_\tau(man)F^{-1}H)(\phi) \\
&= \int_{\mathfrak{q}} \tilde{\pi}_\tau(man)h(X)e^{i\phi(X)}\,dX \\
&= \int_{\mathfrak{q}} \sigma(m)\tau(a)h(\mathrm{Ad}\,(ma)^{-1}X + X_n)\Delta_{k\alpha}^{-1/2}(a)e^{i\phi(X)}\,dX \\
&= \sigma(m)\tau(a)\int_{\mathfrak{q}} h(X + X_n)\exp\left\{i\int(\mathrm{Ad}\,(ma)X)\right\}\Delta_{k\alpha}^{1/2}(a)\,dX \\
&= \sigma(m)\tau(a)\int_{\mathfrak{q}} h(X)\exp\,\{i\phi(\mathrm{Ad}\,(ma)(X - X_n))\}\,\Delta_{k\alpha}^{1/2}(a)\,dX \\
&= \sigma(m)\tau(a)\Delta_{k\alpha}^{1/2}(a)\exp\,\{-i((ma)^{-1}\cdot\phi)(X_n)\}H((ma)^{-1}\cdot\phi).
\end{aligned}
$$

Similarly,

$$
\hat{\pi}_-(man)H(\phi) = \sigma(m)\prod_{i=0}^{j-1} \tau^i(a)\Delta_{k\alpha}^{1/2}(a)\exp\,\{-\,i((ma)^{-1}\cdot\phi)(X_n)\}H((ma)^{-1}\cdot\phi).
$$

Now define B: $L^2(\mathfrak{q}^*, H_\sigma) \longrightarrow L^2(\mathfrak{q}^*, H_\sigma)$ by $g(\phi) \longrightarrow D(\phi)g(\phi)$. Then B is clearly unitary and for $B^{-1}g = G$ we have

$$
\begin{aligned}
(B\hat{\pi}_\tau(man)B^{-1}g)(\phi) &= (B\hat{\pi}_\tau(man)G)(\phi) = D(\phi)\hat{\pi}_\tau(man)G(\phi) \\
&= D(\phi)\sigma(m)\tau(a)\Delta_{k\alpha}^{1/2}(a)\,\exp\,\{-i((ma)^{-1}\cdot\phi)(X_n)\}\;G((ma)^{-1}\cdot\phi) \\
&= D(\phi)\sigma(m)\tau^j(a)\prod_{i=0}^{j-1} \tau^i(a)\Delta_{k\alpha}^{1/2}(a)\,\exp\,\{-i((ma)^{-1}\cdot\phi)(X_n)\}\;G((ma)^{-1}\cdot\phi) \\
&= D((ma)^{-1}\cdot\phi)\prod_{i=0}^{j-1} \tau^i(a)\sigma(m)\Delta_{k\alpha}^{1/2}(a)\exp\,\{-i((ma)^{-1}\cdot\phi)(X_n)\}\;G((ma)^{-1}\cdot\phi) \\
&\qquad\qquad \text{for } \phi \neq 0. \\
&= \sigma(m)\prod_{i=0}^{j-1} \tau^i(a)\Delta_{k\alpha}^{1/2}(a)e^{-i((ma)^{-1}\cdot\phi)(X_n)}g((ma)^{-1}\cdot\phi), \qquad \phi \neq 0 \\
&= \hat{\pi}_-(man)g(\phi), \qquad \phi \neq 0.
\end{aligned}
$$

So $\pi_\tau \simeq \pi_-$ and Theorem 2 has been proven.

Now let G be a connected semisimple Lie group with finite center and Iwasawa decomposition KAN. If M is the centralizer of A in K, then M is compact and $P = MAN$ is a minimal parabolic subgroup of G. Thus $\sigma_1 \otimes \sigma_2 \simeq \Sigma_j\, b_j\tau_j$ where $\sigma_j \in \hat{M} = \hat{M}_d$, the sum is finite, and the b_j's are called the Clebsch-Gordan coefficients for σ_1, σ_2. Theorem 1 now states that

$$
\pi(\sigma_1, \tau_1) \otimes \pi(\sigma_2, \tau_2) \simeq \sum_j^{<\infty} b_j\,\mathrm{Ind}_{MA}^G\,(\sigma_j \times \tau), \qquad \tau = \tau_1\tau_2,
$$

and so the problem of decomposing the tensor product of two principal series representations reduces to that of knowing $\mathrm{Ind}_{MA}^G L$ for all $L \in (MA)^\wedge$. Theorem 2 states that inducing from MA is independent of the character on A and so if we could determine $\mathrm{Ind}_{MA}^G L$ for μ_C-*almost all* $L \in (MA)^\wedge$, then we would know $\mathrm{Ind}_{MA}^G L$ for *all* $L \in (MA)^\wedge$. By Anh's reciprocity theorem, the problem of finding $\mathrm{Ind}_{MA}^G L$ for μ_C-almost all $L \in (MA)^\wedge$ is equivalent to finding $(\pi)_{MA}$ (note that the measure in this decomposition must be absolutely continuous with respect to μ_C) for μ_G-almost all $\pi \in \hat{G}$. We now show to what extent we can solve this last problem when G has real-rank one.

From now on, we assume that G is a connected semisimple real-rank one Lie group with finite center. From Theorems 1 and 2, we know how to decompose the tensor product of two principal series representations of G once we know how to decompose $(\pi)_{MA}$ for μ_G-almost all $\pi \in \hat{G}$, i.e., for almost all principal series and *all* discrete series representations of G.

We now proceed to find $(\pi)_{MA}$ when π is a principal series representation by using Mackey's subgroup theorem.

LEMMA 3. *Let MA act on V by inner automorphism. Then up to a set of measure zero, this action corresponds to the canonical action (on the right) of MA on $P\backslash G$. If $S \subseteq V$ is a cross-section for V/MA, then up to a set of measure zero S also serves as a cross-section for $P\backslash G/MA$.*

PROOF. Let $g = bv$, $b \in P$, $v \in V$ and $ma \in MA$. Then $ma \cdot MANg = MANbvma = NMAvma = NMA(ma)^{-1}v(ma) = MAN(ma)^{-1}v(ma)$ and so the first part is clear. Now define a map Ψ: $V/MA \longrightarrow P\backslash PV/MA$ via Ψ: $\mathcal{O} \longrightarrow D$ where $\mathcal{O} \in V/MA$, D is the double coset containing v, and v is any point in $\mathcal{O}$. That Ψ is well defined and onto is clear. Now suppose $v_1, v_2 \in V$ lie in the same double coset. Then $v_2 = nm'a'v_1ma$ for some $m, m' \in M$, $a, a' \in A$, and $n \in N$. So $v_2 = n(m'a'ma)((ma)^{-1}v_1ma)$ and since elements in $NMAV$ are expressed uniquely, we have $(ma)^{-1}v_1ma = v_2$ and v_1, v_2 lie in the same orbit in V/MA. Thus Ψ is also 1-1 (in fact a Borel isomorphism) and the second part is also clear.

We now fix $X_{-\alpha} \in \mathfrak{g}_{-\alpha} - \{0\}$ and $X_{-2\alpha} \in \mathfrak{g}_{-2\alpha} - \{0\}$.

LEMMA 4. *Let $V' = \{\exp W \exp T = \exp(W + T)\colon\ W \in \mathfrak{g}_{-\alpha}^x,\ T \in \mathfrak{g}_{-2\alpha}^x\}$, and set*

$$S = \begin{cases} \{\exp(\pm X_{-\alpha})\} & \text{if } \dim \mathfrak{g}_{-2\alpha} = 0,\ \dim \mathfrak{g}_{-\alpha} = 1, \\ \{\exp(X_{-\alpha})\} & \text{if } \dim \mathfrak{g}_{-2\alpha} = 0,\ \dim \mathfrak{g}_{-\alpha} > 1, \\ \{\exp(tX_{-\alpha} \pm Y_{-2\alpha})\colon\ t > 0\} & \text{if } \dim \mathfrak{g}_{-2\alpha} = 1, \\ \{\exp(tX_{-\alpha} + Y_{-2\alpha})\colon\ t > 0\} & \text{if } \dim \mathfrak{g}_{-2\alpha} > 1. \end{cases}$$

Then S is a Borel cross-section in V' for the orbits under the action of MA.

PROOF. As discussed in §4, we may assume that G is simple. For the classical simple real-rank one Lie groups it is immediate, by looking at the realizations of these groups and the action of MA on V given in §4, that the above sets form cross-sections for V/MA. That essentially the same arguments hold for the exceptional case, $F_{4,9}$, is not so clear and we thank K. Johnson for forwarding a paper [6] which made clear some of the underlying results in Kostant [9]. So now let $G = F_{4,9}$ and suppose $\exp(t_1 X_{-\alpha} + Y_{-2\alpha})$, $\exp(t_2 X_{-\alpha} + Y_{-2\alpha})$ lie in the same orbit in V/MA. Then there exists $c = ma \in MA$ such that $c^{-1}\exp(t_1 X_{-\alpha} + Y_{-2\alpha})c = \exp(t_2 X_{-\alpha} + Y_{-2\alpha})$. So $a = e$, $m \in M_{Y_{-2\alpha}} = \{m \in M\colon\ m \cdot Y_{-2\alpha} = Y_{-2\alpha}\}$, and $\mathrm{Ad}(m)t_1 X_{-\alpha} = t_2 X_{-\alpha}$, i.e., $\mathrm{Ad}(m)X_{-\alpha} = (t_2/t_1)X_{-\alpha}$ and $t_1 = t_2$. So S meets any orbit in V/MA in at most one point. What we must show is that S does indeed meet each orbit.

We know that $K = \mathrm{Spin}(9)$, $M = \mathrm{Spin}(7)$, $\dim \mathfrak{g}_{-\alpha} = 8$, $\dim \mathfrak{g}_{-2\alpha} = 7$, and that M acts on both $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$. From the lemma of §4, we see that M acts transitively on spheres in both $\mathfrak{g}_{-\alpha}$, $\mathfrak{g}_{-2\alpha}$ and hence that M must act irreducibly on $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$. If we put an $\mathrm{Ad}(M)$-invariant inner product on $\mathfrak{v} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$, then we have that M acts as $\mathrm{SO}(7)$ on $\mathfrak{g}_{-2\alpha}$ and as the Spin respresentation on $\mathfrak{g}_{-\alpha}$ (since these are the only irreducible representations of M in these dimensions). If we let $S_{-\alpha}$, $S_{-2\alpha}$ denote the unit spheres in $\mathfrak{g}_{-\alpha}$, $\mathfrak{g}_{-2\alpha}$ respectively, then we obtain an action of M on $S_{-\alpha} \oplus S_{-2\alpha}$ for which we now show: given (u_1, v_1), $(u_2, v_2) \in S_{-\alpha} \oplus S_{-2\alpha}$, there is an $m \in M$ such that $m \cdot (u_1, v_1) = (u_2, v_2)$. Since $M = \mathrm{Spin}(7)$ is transitive on $S_{-2\alpha}$, we may assume $v_1 = v_2 = v$. The subgroup of $\mathrm{Spin}(7)$ which leaves v fixed is $\mathrm{Spin}(6) \approx \mathrm{SU}(4)$ and when the spin representation of $\mathrm{Spin}(7)$ acting on $\mathfrak{g}_{-\alpha}$ is restricted to $\mathrm{Spin}(6)$, we obtain the irreducible representation of $\mathrm{SU}(4)$ on $\mathbf{C}^4 \approx \mathbf{R}^8$. Since $\mathrm{SU}(4)$ is transitive on S^7, we obtain our result.

Now we know that MA acts transitively on $\mathfrak{g}_{-2\alpha}$ and $M_{Y_{-2\alpha}}A$ acts transitively on $\mathfrak{g}_{-\alpha}$. So let $(W, T) \in \mathfrak{v}$, with $0 \neq W \in \mathfrak{g}_{-\alpha}$, $0 \neq T \in \mathfrak{g}_{-2\alpha}$. Choose $c \in MA$ such that $\mathrm{Ad}(c)T = Y_{-2\alpha}$ and then choose $m \in M_{Y_{-2\alpha}}$ such that $\mathrm{Ad}(m)(\mathrm{Ad}(c)W) = tX_{-\alpha}$, $t > 0$. Then $mc \in MA$ and $\mathrm{Ad}(mc)(W, T) = (tX_{-\alpha}, Y_{-2\alpha})$ as desired.

Note. (1) Since $\mathrm{Ad}(m)tX = t\,\mathrm{Ad}(m)X$, $t \in \mathbf{R}$, $m \in M$, $X \in \mathfrak{v}$, the stability subgroup at any point in $S \subseteq V$ under the action of MA is equal to

$$\{m \in M\colon\ \mathrm{Ad}(m)Y_{-2\alpha} = Y_{-2\alpha},\ \mathrm{Ad}(m)X_{-\alpha} = X_{-\alpha}\} = M_0$$

and so along our cross-section S all the stability subgroups are equal to M_0.

(2) Aside from a set of Haar measure zero in G, $P\backslash G/MA$ can be identified with S and so the following is clear:

LEMMA 5. *MA and MAN are regularly related.*

LEMMA 6. *For $s \in S$, $MA \cap s^{-1}MANs = M_0$.*

PROOF. In light of the correspondence between the action of MA on V by inner automorphism and the natural action of MA on $P\backslash PV$, it suffices to prove the following:

Claim. Suppose H_2 acts on $H_1\backslash G$ on the right. Then $H_1\backslash G$ is the union of orbits O_i. Let $x_i \in O_i$ and M_{x_i} = the isotropy subgroup at x_i. Then $M_{x_i} = H_2 \cap x_i^{-1}H_1x_i$.

Proof of claim.

$$M_{x_i} = \{h \in H_2\colon\ h \cdot (H_1x_i) = H_1x_i\} = \{h \in H_2\colon\ H_1x_ihx_i^{-1} = H_1\}$$
$$= \{h \in H_2\colon\ h \in x_i^{-1}H_1x_i\} = H_2 \cap x_i^{-1}H_1x_i.$$

LEMMA 7. *Let $\eta \in M_v = \{m \in M\colon\ m^{-1}vm = v\}$ for $v \in V$ and let $L \in MA$. Then $\eta \longrightarrow L_{(v\eta v^{-1})}$ is $(L)_{M_v}$.*

PROOF. $\eta \in M_v \Rightarrow v\eta v^{-1} = \eta$ and $L_{(v\eta v^{-1})} = L_{(\eta)}$.
Now define

$$\#(S) = \begin{cases} \text{cardinality of } S, & \text{if the cardinality of } S \text{ is finite,} \\ \infty, & \text{otherwise.} \end{cases}$$

THEOREM 8. *Let S, M_0 be as above, $\sigma \in \hat{M}$, $\tau \in \hat{A}$, and ${}^AR$ denote the regular representation of A. Then*

$$(\pi(\sigma, \tau))_{MA} \simeq \#(S)\,\mathrm{Ind}_{M_0}^{MA}(\sigma)_{M_0} \simeq \#(S)(\mathrm{Ind}_{M_0}^{M}(\sigma)_{M_0} \times {}^AR).$$

PROOF. We apply Mackey's subgroup theorem by identifying the collection of $P{:}MA$ double cosets in G with S and choosing any admissible measure ν on S (note that when $\#(S) = \infty$, there are no orbits of positive measure in V and so ν will be nonatomic). Then

$$(\pi(\sigma, \tau))_{MA} \simeq \int_S^{\oplus} \mathrm{Ind}_{M_0}^{MA}(\sigma)_{M_0}\, d\nu(s) \simeq \#(S)\,\mathrm{Ind}_{M_0}^{MA}(\sigma)_{M_0}$$
$$\simeq \#(S)\,\mathrm{Ind}_{M_0\times\{e\}}^{M\times A}(\sigma)_{M_0} \times 1 \simeq \#(S)(\mathrm{Ind}_{M_0}^{M}(\sigma)_{M_0} \times \mathrm{Ind}_{\{e\}}^{A} 1)$$
$$\simeq \#(S)(\mathrm{Ind}_{M_0}^{M}(\sigma)_{M_0} \times {}^AR).$$

Note that in determining $\mathrm{Ind}_{M_0}^{M}(\sigma)_{M_0}$, we may use the compact version of the reciprocity theorem and that the multiplicity of a $\sigma' \times \tau' \in MA$ occurring in $(\pi(\sigma, \tau))_{MA}$ does not depend on τ'. So the above decomposition can be written

$$(\pi(\sigma, \tau))_{MA} \simeq \#(S) \int_{\hat{C}}^{\oplus} n(\pi, \sigma')(\sigma' \times \tau')\, d\mu_C(\sigma', \tau')$$

where μ_C denotes Plancherel measure on $MA = \hat{C}$ and $n(\dot{\pi}, \sigma')$ can be explicitly computed (see Theorem 16).

Now let $\hat{G}_d$ denote the discrete series representations for G. We now attempt to find $(\pi)_{MA}$ for $\pi \in \hat{G}_d$. We first find $(\mathrm{Ind}_K^G\, \sigma)_{MA}$ for $\sigma \in \hat{K}$.

LEMMA 9. *The action of M on N by inner automorphism corresponds to the action of MA on $K\backslash G$ on the right and if $S_1 \subseteq N$ denotes a cross-section for N/M, then S_1 also serves as a cross-section for $K\backslash G/MA$.*

PROOF. Let $g = kna$, $k \in K$, $a \in A$, $n \in N$ and $ma' \in MA$. Then $ma' \cdot Kg = Knama' = Km^{-1}nmaa'$ and so the first part is clear. Define Ψ: $N/M \longrightarrow K\backslash G/MA$ by sending an orbit $\mathcal{O}$ to the double coset D, where D is the double coset containing v, and v is any point in $\mathcal{O}$. Two points in the same orbit in N/M clearly lie in the same $K:MA$ double coset and so Ψ is well defined. That Ψ is onto is clear. Now suppose n_1, n_2 lie in the same $K:MA$ double coset. Then $n_2 = kn_1ma$ for some $k \in K$, $ma \in MA$. Thus $n_2 = (km)(m^{-1}n_1m)a$ and since elements in KNA are expressed uniquely, we have $m^{-1}n_1m = n_2$, i.e., n_1, n_2 are in the same orbit in N/M and Ψ is 1-1. Lemma 9 is now clear.

LEMMA 10. *Let $X_\alpha = \theta X_{-\alpha} \in \mathfrak{g}_\alpha - \{0\}$, $Y_{2\alpha} = \theta Y_{-2\alpha} \in \mathfrak{g}_{2\alpha} - \{0\}$, and set*

$$S_1 = \begin{cases} \{\exp(tX_\alpha):\ t \in \mathbf{R}\} & \text{if } \dim \mathfrak{g}_{2\alpha} = 0,\ \dim \mathfrak{g}_\alpha = 1, \\ \{\exp(tX_\alpha):\ t \geqslant 0\} & \text{if } \dim \mathfrak{g}_{2\alpha} = 0,\ \dim \mathfrak{g}_\alpha > 1, \\ \{\exp(tX_\alpha + sY_\alpha):\ t \geqslant 0,\ s \in \mathbf{R}\} & \text{if } \dim \mathfrak{g}_{2\alpha} = 1, \\ \{\exp(tX_\alpha + sY_\alpha):\ t \geqslant 0,\ s \geqslant 0\} & \text{if } \dim \mathfrak{g}_{2\alpha} > 1. \end{cases}$$

Then S_1 is a Borel cross-section in N for the orbits under the action of M.

PROOF. We proceed as in Lemma 4. Since the classical cases are again clear by inspection, we shall deal only with the exceptional case. First suppose $\exp(t_1X_\alpha + s_1Y_\alpha)$, $\exp(t_2X_\alpha + s_2Y_\alpha)$ lie in the same orbit in N/M. Then there exists $m \in M$ such that $\mathrm{Ad}(m)t_1X_\alpha = \mathrm{Ad}(m)t_2X_\alpha$, $\mathrm{Ad}(m)s_1Y_\alpha = \mathrm{Ad}(m)s_2Y_\alpha$ where t_1, t_2, s_1, s_2 are all $\geqslant 0$. Thus $t_1 = t_2$, $s_1 = s_2$, and S_1 meets any orbit in at most one point.

As in Lemma 4, M acts transitively on spheres in $\mathfrak{g}_{2\alpha}$ while M_{Y_α} acts transitively on spheres in $\mathfrak{g}_\alpha$. So for $(W, T) \in \mathfrak{n} = LA(N)$ we may choose $m \in M$ such that $\mathrm{Ad}(m)T = sY_\alpha$ for some $s \geqslant 0$ and then an $m' \in M_{Y_\alpha}$ such that $\mathrm{Ad}(m')\mathrm{Ad}(m)W = tX_\alpha$ for some $t \geqslant 0$. Then $m'm \in M$ and $\mathrm{Ad}(m'm)(W, T) = (tX_\alpha, sY_\alpha)$ and hence S_1 meets every orbit in N/M.

Note. (1) The isotropy subgroup along S_1 are again all equal, and in each case we have $\#(S_1) = \infty$.

(2) Since $\mathrm{Ad}(m)\theta X_\alpha = \theta(\mathrm{Ad}\, mX_\alpha)$ for $m \in M$ and $X_\alpha \in \mathfrak{g}_{k\alpha}$, $k = 1, 2$, we have that $\{m \in M\colon \mathrm{Ad}(m)X_\alpha = X_\alpha\} = \{m \in M\colon \mathrm{Ad}(m)\theta X_\alpha = \theta X_\alpha\}$. So $M_0 = \{m \in M\colon \mathrm{Ad}(m)X_{-\alpha} = X_{-\alpha},\ \mathrm{Ad}(m)Y_{-2\alpha} = Y_{-2\alpha}\} = \{m \in M\colon \mathrm{Ad}(m)X_\alpha = X_\alpha,\ \mathrm{Ad}(m)Y_\alpha = Y_\alpha\} =$ the isotropy subgroup (s) along S_1.

(3) We may identify S_1 with $K\backslash G/MA$ and so the following is clear.

LEMMA 11. *K and MA are regularly related in G.*

LEMMA 12. *For $s \in S_1$, $MA \cap s^{-1}Ks = M_0$.*

PROOF. Same as claim in Lemma 6.

THEOREM 13. *Let S_1, M_0 be as above and $\sigma \in \hat{K}$. Then*

$$(\mathrm{Ind}_K^G\ \sigma)_{MA} \simeq \infty(\mathrm{Ind}_{M_0}^M (\sigma)_{M_0} \times {}^AR).$$

PROOF. Mackey's subgroup theorem implies that

$$(\mathrm{Ind}_K^G\ \sigma)_{MA} \simeq \int_{S_1}^{\oplus} \mathrm{Ind}_{M_0}^{MA}\ (\sigma)_{M_0} d\nu(s)$$

where ν is any admissible measure on S_1. So as before

$$(\mathrm{Ind}_K^G\ \sigma)_{MA} \simeq \#(S_1)(\mathrm{Ind}_{M_0}^M (\sigma)_{M_0} \times {}^AR) \simeq \infty(\mathrm{Ind}_{M_0}^M (\sigma)_{M_0} \times {}^AR).$$

LEMMA 14. *$\pi_0 \in \hat{G}_d \Rightarrow \exists \delta \in \hat{K} \ni \pi_0$ is contained in $\mathrm{Ind}_K^G\ \delta$.*

PROOF. We use Ahn's reciprocity theorem. We know that, for all $\pi \in \hat{G}$, $(\pi)_K \simeq \Sigma_{\delta\in\hat{K}}\, n(\pi, \delta)\delta$ and so, for all $\delta \in \hat{K}$, we have $\mathrm{Ind}_K^G\ \delta \simeq \int_{\hat{G}}^{\oplus} n(\pi, \delta)\pi\, d\mu_G(\pi)$. Now there exists a $\delta \in \hat{K}$ for which $n(\pi_0, \delta) \neq 0$ (lest $(\pi_0)_K = 0$) and since $\mu_G(\{\pi_0\}) > 0$, we have found a $\delta \in \hat{K}$ for which π_0 is contained in $\mathrm{Ind}_K^G\ \delta$.

THEOREM 15. *$\pi \in \hat{G}_d \Rightarrow (\pi)_{MA} \simeq \int_{\hat{C}}^{\oplus} n(\sigma, \tau, \pi)(\sigma \times \tau)d\mu_C(\sigma, \tau)$ where μ_C is Plancherel measure on $\hat{C} = MA - (MA)^\wedge$ and $n(\sigma, \tau, \pi) \in \{0, 1, 2, \cdots, \infty\}$.*

PROOF. By Lemma 14, $\exists \delta \in \hat{K}$ such that π is contained in $\mathrm{Ind}_K^G\ \delta$ and by Theorem 13

$$(\mathrm{Ind}_K^G\ \delta)_{MA} \simeq \int_{\hat{C}}^{\oplus} n'(\sigma, \delta, \pi)(\sigma \times \tau)d\mu_C(\sigma, \tau)$$

where $n'(\sigma, \delta, \pi) \in \{0, \infty\}$. So by [4, p. 273] we have

$$(\pi)_{MA} \simeq \int_{\hat{C}}^{\oplus} n''(\sigma, \tau, \delta, \pi)(\sigma \times \tau)d\nu(\sigma, \tau)$$

where $\nu \ll \mu_C$ and $n''(\sigma, \tau, \delta, \pi) \leqslant n'(\sigma, \tau, \delta, \pi)$ for ν almost all (σ, τ). We may now write $n''(\sigma, \tau, \delta, \pi) = n''(\sigma, \tau, \pi)$ since, if π is also contained in $\mathrm{Ind}_K^G\ \delta'$, $\delta' \in \hat{K}$, then we would have

$$\int_{\hat{C}}^{\oplus} n''(\sigma, \tau, \delta, \pi)(\sigma \times \tau)d\nu(\sigma, \tau) \simeq (\pi)_{MA} \simeq \int_{\hat{C}}^{\oplus} n_2''(\sigma, \tau, \delta', \pi)(\sigma \times \tau)d\nu_2(\sigma, \tau)$$

and hence that $\nu \sim \nu_2$ and $n'' = n_2''$ for ν almost all (σ, τ). If we now choose a μ_C-measurable set $E \subseteq \hat{C}$ such that ν and $\chi_E \cdot \mu_C$ (χ_E is the characteristic function of E) have the same null sets, then by [12, p. 123] we obtain

$$(\pi)_{MA} \simeq \int_{\hat{C}}^{\oplus} n''(\sigma, \tau, \pi)(\sigma \times \tau)d(\chi_E \cdot \mu_C)(\sigma, \tau).$$

If we now define

$$n(\sigma, \tau, \pi) = \begin{cases} n''(\sigma, \tau, \pi) & \text{if } (\sigma, \tau) \in E, \\ 0 & \text{otherwise,} \end{cases}$$

then we have $(\pi)_{MA} \simeq \int_{\hat{C}}^{\oplus} n(\sigma, \tau, \pi)(\sigma \times \tau)d\mu_C(\sigma, \tau)$ and the proof is complete.

Combining Theorems 8 and 15, we now see that for μ_G almost all $\pi \in \hat{G}$

$$(\pi)_{MA} \simeq \int_{\hat{C}}^{\oplus} n(\sigma, \tau, \pi)(\sigma \times \tau)d\mu_C(\sigma, \tau)$$

where $n(\sigma, \tau, \pi)$ is a measurable function on $\hat{M} \times \hat{A} \times \hat{G}$. So by Anh's reciprocity theorem and Theorem 2, we have for *all* $(\sigma, \tau) \in (MA)^\wedge$

$$\operatorname{Ind}_{MA}^{G} \sigma \times \tau \simeq \int_{\hat{G}}^{\oplus} n(\sigma, \tau, \pi)\pi\, d\mu_G(\pi)$$

where $n(\sigma, \tau, \pi)$ can be computed explicitly for π in the principal series and $n(\sigma, \tau, \pi) \in \{0, 1, 2, \cdots, \infty\}$ for $\pi \in \hat{G}_d$. Note that since $\operatorname{Ind}_{MA}^{G} \sigma \times \tau$ is independent of $\tau \in \hat{A}$, we may now conclude that $n(\sigma, \tau, \pi) = n(\sigma, \pi)$.

THEOREM 16. *Let G be a connected semisimple real-rank one Lie group with finite center, let $\pi(\sigma_1, \tau_1)$, $\pi(\sigma_2, \tau_2)$ be two principal series representations, let $\sigma_1 \otimes \sigma_2 = \Sigma_{i=1}^{n} b_j\sigma_j$, let M_0 be as in Lemma* 6, *and for $\chi_1, \chi_2 \in$* Rep(M_0) *let $I(\chi_1, \chi_2)$ denote the intertwining number for χ_1 and χ_2. Then $\pi(\sigma_1, \tau_1) \otimes \pi(\sigma_2, \tau_2) \simeq T_c \oplus T_d$ where T_c is a continuous direct integral with respect to Plancherel measure on $\hat{G}$ of representations from the principal series of G and T_d is a discrete direct sum of discrete series representations. If* dim $\mathfrak{g}_{-2\alpha} \neq 0$, *then the multiplicity of $\pi(\sigma, \tau)$ in T_c is either* 0 *or* ∞ *according to whether $\Sigma_{j=1}^{n} I((\sigma)_{M_0}, (\sigma_j)_{M_0})$ is* 0 *or not. If* dim $\mathfrak{g}_{-2\alpha} = 0$, *then the multiplicity of $\pi(\sigma, \tau)$ in T_c is finite and equals $\epsilon \cdot \Sigma_{j=1}^{n} I((\sigma)_{M_0}, (\sigma_j)_{M_0})b_j$ where $\epsilon = 2$ if* dim $\mathfrak{g}_\alpha = 1$ *and $\epsilon = 1$ if* dim $\mathfrak{g}_\alpha > 1$.

PROOF. All but the multiplicity of $\pi(\sigma, \tau)$ in T_c is clear. Recall that

$$\pi(\sigma_1, \tau_1) \otimes \pi(\sigma_2, \tau_2) \simeq \sum_{j=1}^{n} b_j \operatorname{Ind}_{MA}^{G} \sigma_j \times \tau', \qquad \tau' = \tau_1\tau_2,$$

and that the multiplicity of $\pi(\sigma, \tau)$ in $\operatorname{Ind}_{MA}^{G} \sigma_j \times \tau'$ equals the multiplicity of $\sigma_j \times \tau'$ in $(\pi(\sigma, \tau)_{MA} \simeq \#(S)(\operatorname{Ind}_{M_0}^{M}(\sigma)_{M_0} \times {}^{A}R)$, viz., $\#(S) \cdot$ (the multiplicity of σ_j in $\operatorname{Ind}_{M_0}^{M}(\sigma)_{M_0}$).

Now let

$$(\sigma)_{M_0} = \sum_{\rho \in \hat{M}_0} n(\sigma, \rho)\rho, \qquad (\sigma_j)_{M_0} = \sum_{\rho' \in \hat{M}_0} n(\sigma_j, \rho')\rho',$$

and

$$\operatorname{Ind}_{M_0}^{M} (\sigma)_{M_0} = \sum_{\nu \in \hat{M}} \alpha_\nu \nu.$$

Then by compact reciprocity, the multiplicity of σ_j in $\operatorname{Ind}_{M_0}^{M}(\sigma)_{M_0}$ is given by:

$$\alpha_{\sigma_j} = n(\operatorname{Ind}_{M_0}^{M} (\sigma)_{M_0}, \sigma_j) = \sum_{\rho \in \hat{M}_0} n(\sigma, \rho) n(\operatorname{Ind}_{M_0}^{M} \rho, \sigma_j)$$

$$= \sum_{\rho \in \hat{M}_0} n(\sigma, \rho) n((\sigma_j)_{M_0}, \rho) = \sum_{\rho \in \hat{M}_0} n(\sigma, \rho) n(\sigma_j, \rho) = I((\sigma)_{M_0}, (\sigma_j)_{M_0}).$$

So the multiplicity of $\pi(\sigma, \tau)$ in $\pi(\sigma_1, \tau_1) \otimes \pi(\sigma_2, \tau_2)$ will be

$$\#(S) \cdot \sum_{j=1}^{n} I((\sigma)_{M_0}, (\sigma_j)_{M_0}) b_j$$

and the theorem is then clear from Lemma 4.

Note that even though we are not able to give the multiplicity of a discrete series representation occurring in T_d at this time, Theorem 13 does provide us with some information about discrete series representations which do *not* occur in T_d.

VI. Examples.

A. *G complex.* When G is a connected complex semisimple Lie group, we may use the techniques of §5 to simplify some of the recent work of Floyd Williams [18]. When G is complex, MA is a Cartan subgroup and the principal series $\{\pi(\sigma, \tau)\colon \sigma \in \hat{M}, \tau \in \hat{A}\}$ consitute almost all of $\hat{G}$. So by Theorems 1 and 2 of §5 and Anh's reciprocity theorem, we know how to decompose $\pi(\sigma_1, \tau_1) \otimes \pi(\sigma_2, \tau_2) \simeq \operatorname{Ind}_{MA}^{G} (\sigma_1\sigma_2)(\tau_1\tau_2)$ once we know $(\pi(\sigma, \tau))_{MA}$ for almost all $(\sigma, \tau) \in \hat{M} \times \hat{A}$.

Let $\{\alpha_1, \cdots, \alpha_l\}$ denote the simple roots of $\mathfrak{g}$, $\{\alpha_1, \cdots, \alpha_l, \alpha_{l+1}, \cdots, \alpha_k\}$ the positive roots, $X_j \in \mathfrak{g}_{-\alpha_j} - \{0\}$,

$$V_0 = \left\{ \prod_{j=1}^{k} \exp z_j X_j \in V\colon z_j \in \mathbf{C}, z_j \neq 0 \text{ for } 1 \leq j \leq l \right\},$$

and

$$V' = \left\{ \prod_{j=1}^{k} \exp z_j X_j \in V: \ z_j = 0, \ 1 \leq j \leq l \right\}.$$

Then $V - V_0$ has Haar measure zero and Williams shows that V_0/MA can be identified with $V' (\approx \mathbf{C}^{k-l})$. Choosing $S = V'$, we see that S is either one point or infinite depending upon whether N is abelian (if and only if all positive roots are simple) or not and that up to a set of Haar measure zero in G, S may be identified with $MAN \backslash G/MA$. For $v \in S$, let $M_v = \{c \in MA: \ c^{-1}vc = v\}$. Then:

LEMMA. *$M_v = Z(G)$, the center of G, for each $v \in S$.*

PROOF. Clearly $Z(G) \subseteq M_v$ for each $v \in S$. So let $c \in M_v$ where $v = \exp X$, $X = \Sigma_{\alpha > 0} X_{-\alpha}$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha} - \{0\}$. If $\exp H = c$, then $\alpha(H) = 0$ for all α [in fact $m_X = \{H \in h: \ H \cdot X = X\} = \{H \in h: \ \Sigma\, e^{\alpha(H)} X_{-\alpha} = \Sigma\, X_{-\alpha}\} = \{H \in h: \ \alpha(H) = 0, \ \text{all } \alpha\} = \{0\}$]. Let $g = \exp(h_0 + \Sigma\, X_\alpha)$. Then

$$\begin{aligned} c^{-1}gc &= \exp\left(e^{\operatorname{ad} H} h_0 + \sum e^{\operatorname{ad} H} X_\alpha\right) \\ &= \exp\left(h_0 + \sum e^{\alpha(H)} X_\alpha\right) = \exp\left(h_0 + \sum X_\alpha\right) = g, \end{aligned}$$

i.e., $c \in Z(G)$. Thus by applying Mackey's subgroup theorem, we obtain

$$(\pi(\sigma, \tau))_{MA} \simeq \int_S^{\oplus} \operatorname{Ind}_{M_v}^{MA} (\sigma \times \tau)_{M_v} \, d\mu(v) \simeq \epsilon \cdot \operatorname{Ind}_{Z(G)}^{MA} (\sigma)_{Z(G)}$$

where $\epsilon = 1$ or ∞ according to whether N is abelian or not. Note that $(\sigma)_{Z(G)}$ is irreducible for $\sigma \in \hat{M}$. Hence we have the following:

THEOREM (WILLIAMS, 1972). *Let G be a connected complex semisimple Lie group and MA a Cartan subgroup of G. Suppose $\pi(\sigma_1, \tau_2)$, $\pi(\sigma_2, \tau_2)$ are two principal series representations where $\sigma_1, \sigma_2 \in \hat{M}$, $\tau_1, \tau_2 \in \hat{A}$. Then*

$$\pi(\sigma_1, \tau_1) \otimes \pi(\sigma_2, \tau_2) \simeq \int_{\hat{G}}^{\oplus} \epsilon \cdot \pi(\sigma, \tau) \, d\mu_G(\pi)$$

where the elements $\pi = \pi(\sigma, \tau)$ occurring in this decomposition are precisely those for which σ and $\sigma_1\sigma_2$ coincide on $Z(G)$ and $\epsilon = 1$ or ∞ according to whether N is abelian or not.

B. $G = \mathrm{SL}(2, \mathbf{R})$. When $G = \mathrm{SL}(2, \mathbf{R})$, our techniques yield a complete solution to the problem of decomposing the tensor product of two principal series representations, since we are able to compute $(\pi)_{MA}$ for $\pi \in \hat{G}_d$. Recall that

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, a, b, c, d \in \mathbf{R} \right\}, \qquad K = \mathrm{SO}(2),$$

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\} \approx (0, \infty), \qquad V = \left\{ \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} : v \in \mathbf{R} \right\},$$

$$M = Z(G) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix},$$

$\hat{A} \approx \mathbf{R}$, and $\hat{M} = \{\sigma^{\pm}\}$. The discrete series for $\mathrm{SL}(2, \mathbf{R})$ (see [16]) can be parameterized by nonzero half-integers and briefly described as follows: On the upper half plane P, for $n = 1, 3/2, 2, \cdots$ (resp. $n = -1, -3/2, -2, \cdots$), we take the Hilbert space $H_{2,n}(P)$ of holomorphic (resp. conjugate holomorphic) functions on P with the inner product

$$(f, g)_n = \frac{1}{\Gamma(2n-1)} \int_P f(x+iy)\overline{g(x+iy)}\, y^{-2+2|m|}\, dx\, dy$$

while for $n = ½$ (resp. $n = -½$), $H_{2,½}(P)$ (resp. $H_{2,-½}(P)$) is the space of holomorphic (resp. conjugate holomorphic) functions on P with the property that $\int_{-\infty}^{\infty} |f(x+iy)|^2\, dx$ is bounded uniformly in y for $y > 0$. In this case, f has boundary values almost everywhere on the real axis and if $f(x)$, $g(x)$ denote the boundary values of $f(z)$, $g(z)$, then the inner product is given by $\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx$. For $n = \{\pm 1/2, \pm 1, \pm 3/2, \cdots\}$, we have a representation $D_n \in \hat{G}$ acting on $H_{2,n}$ via

$$D_n(g)f(z) = (bz + d)^{-2n} f\left(\frac{az + c}{bz + d}\right), \qquad n = 1/2, 1, 3/2, \cdots,$$

$$D_n(g)f(z) = (b\bar{z} + d)^{-2|n|} f\left(\frac{az + c}{bz + d}\right), \qquad n = -1/2, -1, -3/2, \cdots,$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The discrete series of $G = \mathrm{SL}(2, \mathbf{R})$ is the family of representations $\hat{G}_d = \{D_n : n = \pm 1, \pm 3/2, \pm 2, \cdots\}$. The representations $D_{½}, D_{-½}$ are not square-integrable and it is known that $D_{½} \oplus D_{-½} \simeq \pi(\sigma^-, 0)$. For $\tau \in A \approx (0, \infty)$, let $d\mu(t) = dt/t$ denote Haar measure on A and $L^2(A) = L^2((0, \infty); d\mu)$. For $a, t \in A$, $f \in L^2(A)$, let $\rho(a)f(t) = f(at)$ be the regular representation of A and $\tilde{\rho}(a)f(t) = f(a^2 t)$. Then

LEMMA. $\rho \simeq \tilde{\rho}$.

PROOF. Let $\Psi: L^2(A) \longrightarrow L^2(A)$ by $\Psi(f(t)) = 2^{-½} f(\sqrt{t})$, $t \in (0, \infty)$. Then

$$\|\Psi f\|^2 = \int_0^\infty |\Psi f(t)|^2 \frac{dt}{t} = \frac{1}{2}\int_0^\infty |f(\sqrt{t})|^2 \frac{dt}{t}$$

$$= \frac{1}{2}\int_0^\infty |f(t)|^2 \frac{2tdt}{t^2} = \int_0^\infty |f(t)|^2 \frac{dt}{t} = \|f\|^2$$

and for $F = \Psi^{-1} f$

$$(\Psi\rho(a)\Psi^{-1}f)(t) = 2^{-1/2}\rho(a)F(\sqrt{t}) = 2^{-1/2}F(a\sqrt{t})$$

$$= f(a^2 t) = (\tilde{\rho}(a)f)(t).$$

LEMMA.

$$(D_n)_{MA} \simeq \begin{cases} \sigma^+ \times \rho & \text{if } n = \pm 1, \pm 2, \cdots, \\ \sigma^- \times \rho & \text{if } n = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots. \end{cases}$$

PROOF. Let $n = 1/2, 1, 3/2, \cdots$ and

$$g = ma = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \in MA.$$

As in [16], for $n > 1/2$ we let $H_{(2,n)^\wedge}(P)$ be the Hilbert space of functions on $(0, \infty)$ with the inner product

$$(\phi, \Psi)_n = \frac{1}{2^{2n-1}}\int_0^\infty \phi(t)\overline{\Psi(t)}t^{-2n+1}\,dt$$

and define the inverse Fourier-Laplace transform of a holomorphic function f in $H_{2,n}(P)$ by

$$(\digamma f)(t) = (2\pi)^{-1/2}\int_{-\infty}^\infty f(x+iy)e^{-it(x+iy)}\,dx.$$

This integral is independent of $y > 0$, $\digamma f(t)$ vanishes almost everywhere for $t < 0$, and $\digamma\colon H_{2,n}(P) \longrightarrow H_{(2,n)^\wedge}(P)$ is an isometric isomorphism.

For $n = 1/2$, the inverse Fourier-Laplace transform on $H_{2,1/2}(P)$ reduces to the ordinary Fourier transform on $L_2(\mathbf{R})$, i.e.,

$$\digamma(f)(t) = (2\pi)^{-1/2}\int_{-\infty}^\infty f(x)e^{-itx}\,dx$$

and, as is well-known, $\digamma f(t)$ vanishes almost everywhere for $t < 0$. Setting $H_{(2,1/2)^\wedge}(P) = L_2((0,\infty), dt)$, we have that $\digamma$ is an isometric isomorphism from $H_{2,1/2}(P)$ to $H_{(2,1/2)^\wedge}(P)$.

For $f \in H_{2,n}(P)$, $n = 1/2, 1, 3/2, \cdots$, and $g = ma \in MA$, we have $D_n(g)f(z) = (ma)^{-2n} f(a^{-2}z)$. We now use $\digamma$ to realize $(D_n)_{MA}$ on $H_{(2,n)^\frown}(P)$: $(\digamma D_n(g)\digamma^{-1}g)(t) = (\digamma D_n(g)f)(z)$ where $\digamma^{-1}g = f$.

$$= (2\pi)^{-1/2} \int_{\mathbf{R}} (ma)^{-2n} f(a^{-2}z) e^{-itz}\, dx$$

(with the obvious changes for $n = 1/2$)

$$= (2\pi)^{-1/2} \int_{\mathbf{R}} (ma)^{-2n} f(z) e^{-ita^2 z} a^2\, dx$$

$$= m^{-2n} a^{2-2n} (2\pi)^{-1/2} \int_{\mathbf{R}} f(z) e^{-ita^2 z}\, dx$$

$$= m^{-2n} a^{2-2n} g(a^2 t).$$

Set $(W(n))^\wedge f(t) = (2t)^{-n+1/2} f(t)$. Then $(W(n))^\wedge$ is a unitary mapping from $H_{(2,n)^\frown}(P)$ to $H_{(2,1/2)^\frown}(P)$ and we may now realize $(D_n)_{MA}$ on $H_{(2,1/2)^\frown}(P)$:

$$(\widehat{W(n)} D_n(g) \widehat{W(n)}^{-1} f)(t) = \widehat{W(n)} m^{-2n} a^{2-2n} F(a^2 t)$$

$$= (2t)^{-n+1/2} m^{-2n} a^{2-2n} F(a^2 t)$$

$$= (2t)^{-n+1/2} m^{-2n} a^{2-2n} (2a^2 t)^{n-1/2} f(a^2 t)$$

$$= m^{-2n} a f(a^2 t).$$

Now set $\Phi(f(t)) = t^{1/2} f(t)$. Then Φ is a unitary mapping of $H_{(2,1/2)^\frown}(P) = L^2((0, \infty), dt)$ into $L^2((0, \infty), dt/t) = L^2(A)$ and

$$(\Phi D_n(g) \Phi^{-1} f)(t) = (\Phi D_n(g) F)(t) = t^{1/2} m^{-2n} a F(a^2 t)$$
$$= t^{1/2} m^{-2n} a a^{-1} t^{-1/2} f(a^2 t) = m^{-2n} f(a^2 t).$$

For $n = 1, 2, \cdots$, this last representation is equivalent to $\sigma^+ \times \rho$ while for $n = 1/2, 3/2, \cdots$, this last representation is equivalent to $\sigma^- \times \rho$. A similar argument holds for $n = -1/2, -1, -3/2, \cdots$ and so the lemma is now clear.

Now recall that MA acts on V in two nonzero orbits and so $(\pi(\sigma, \tau))_{MA} \simeq 2\,\mathrm{Ind}_M^{MA}\, \sigma \simeq 2(\sigma \times \rho)$ for $\sigma \in \hat{M}$, $\tau \in \hat{A}$. Thus we have

THEOREM (PUKANSZKY, 1960). *Let* $\pi(\sigma_1, \tau_1)$, $\pi(\sigma_2, \sigma_2)$, $\sigma_1, \sigma_2 \in \hat{M} = \{\sigma^\pm\}$, $\tau_1, \tau_2 \in \hat{A}$ *be two representations from the principal series of* $G = \mathrm{SL}(2, \mathbf{R})$. *Then*

$$\pi(\sigma_1, \tau_1) \otimes \pi(\sigma_2, \tau_2) \simeq \begin{cases} 2\int_{\hat{G}}^{\oplus} \pi(\sigma^+, \tau)\, d\mu_G(\pi) \oplus \sum_{n=\pm 1, \pm 2, \cdots} \oplus D_n & \text{if } \sigma_1\sigma_2 = \sigma^+, \\ 2\int_{\hat{G}}^{\oplus} \pi(\sigma^-, \tau)\, d\mu_G(\pi) \oplus \sum_{n=\pm 3/2, \cdots} \oplus D_n & \text{if } \sigma_1\sigma_2 = \sigma^-. \end{cases}$$

C. $G \approx SO_e(3, 1) \approx SL(2, \mathbf{C})/\{\pm e\}$. When $G = SO_e(3, 1)$, S is one point, $M_0 = \{e\}$, $(\pi(\sigma, \tau))_{MA} \simeq \mathrm{Ind}_{\{e\}}^{MA} 1 \simeq {}^{MA}R$ for all $(\sigma, \tau) \in \hat{M} \times \hat{A}$, and there is no discrete series for G. Thus we obtain:

THEOREM (MACKEY, 1952-NAĬMARK, 1958). *Let* $\pi(\sigma_1, \tau_1)$, $\pi(\sigma_2, \tau_2)$ *where* $\sigma_1, \sigma_2 \in \hat{M} = SO(2)^\wedge$, $\tau_1, \tau_2 \in \hat{A}$, *be two representations from the principal series of* $G = SO_e(3, 1)$. *Then*

$$\pi(\sigma_1, \tau_1) \otimes \pi(\sigma_2, \tau_2) \simeq \int_{\hat{G}}^{\oplus} \pi(\sigma, \tau)\, d\mu_{\hat{G}}(\pi).$$

Note that Theorem 16 also yields a complete solution to the decomposition of the tensor product of two principal series representations for $G = SO_e(2n + 1, 1)$, $n \geqslant 1$, since in this case $\hat{G}_d = \phi$.

D. $G = SL(n, \mathbf{R})$, $n \geqslant 3$. We shall now show how the techniques developed in §5 yield a complete solution to the problem of decomposing the tensor product of two (minimal) principal series representations of $G = SL(n, \mathbf{R})$, $n \geqslant 3$. Let $D(a_1, \cdots, a_n)$ denote the diagonal matrix in G with entries $a_1, \cdots, a_n$ and let $e_i(D(a_1, \cdots, a_n)) = a_i$, $i = 1, \cdots, n$. We have that

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbf{R}) \colon \operatorname{tr}(X) = 0\}, \quad \mathfrak{k} = \{X \in \mathfrak{g} \colon X^t = -X\},$$

$$\mathfrak{p} = \{X \in \mathfrak{g} \colon X^t = X\}, \qquad \mathfrak{a} = \Big\{D(a_1, \cdots, a_n) \colon a_i \in \mathbf{R}, \sum_{i=1}^n a_i = 0\Big\},$$

θ is negative transpose, $\Delta = \{\pm(e_i - e_j) \colon 1 \leqslant i \leqslant j \leqslant n\}$, $\Delta^+ = \{e_i - e_j \colon 1 \leqslant i \leqslant j \leqslant n\}$, $\mathfrak{g}_{e_i - e_j} = \mathbf{R} \cdot E_{ij}$ (E_{ij} is the $n \times n$ matrix consisting of all zeros and a single one in the ijth place), the simple roots are $\{e_i - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n\}$, $K = SO(n)$, A is the subgroup of all diagonal matrices with positive diagonal elements, N is the subgroup of all matrices which are one on the diagonal and zero below and M is the subgroup of all matrices which are ± 1 on the diagonal and zero elsewhere. For n odd, the center of G is $Z(G) = \{e\}$ while for n even, $Z(G) = \{\pm e\}$. For n even, we put $(Z(G))^\wedge = \{\chi^+, \chi^-\}$. The group of all diagonal matrices in G, MA, is a Cartan subgroup of G and the principal series of G is the family of representations $\{\pi_0(\sigma, \tau) = \mathrm{Ind}_{MAN}^G\, \sigma \times \tau \colon \sigma \in \hat{M},\ \tau \in \hat{A}\}$.

For $n \geqslant 3$, Rank $G \neq$ Rank K and so there is no discrete series to contend with. However, G has Cartan subgroups not conjugate to MA and for each conjugacy class there is a cuspidal parabolic subgroup and a corresponding family of continuous series representations, F, for which $\mu_G(F) \neq 0$. Let $P_0 = MAN$, a minimal parabolic subgroup of G. We now construct a complete set of nonassociate cuspidal parabolic subgroups of G. If t is the largest integer such that $2t \leqslant n$, then there are exactly t mutually nonconjugate Cartan subgroups which are not conjugate to MA [17, vol. I, p. 95] and so there are $t+1$ nonassociate cuspidal parabolic subgroups for G. The following construction is detailed in [17, vol. I, pp. 66–78]. Let $\theta_1 = \{e_1 - e_2\}$, $\theta_2 = \{e_1 - e_2, e_3 - e_4\}, \cdots, \theta_t = \{e_1 - e_2, e_3 - e_4, \cdots, e_{2t-1} - e_{2t}\}$. Let $\langle\theta_i\rangle$ denote the set of those $\lambda \in \Delta$ which are linear combinations of the elements in θ_1, $\langle\theta_i\rangle^\pm = \Delta^\pm \cap \langle\theta_i\rangle$, and $\mathfrak{n}^\pm(\theta_i) = \Sigma_{\lambda\in\langle\theta_i\rangle^\pm} \mathfrak{g}_\lambda$ (note that $\langle\theta_i\rangle^+ = \theta_i$). If $\mathfrak{g}(\theta_i)$ denotes the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}^+(\theta_i) + \mathfrak{n}^-(\theta_i)$, then $\mathfrak{g}(\theta_i) = \mathfrak{k}(\theta_i) + \mathfrak{p}(\theta_i)$ where $\mathfrak{k}(\theta_i) = \mathfrak{g}(\theta_i) \cap \mathfrak{k}$, $\mathfrak{p}(\theta_i) = \mathfrak{g}(\theta_i) \cap \mathfrak{p}$. Furthermore, $\mathfrak{g}(\theta_i)$ is semisimple and $\mathfrak{g}(\theta_i) = \mathfrak{k}(\theta_i) + \mathfrak{p}(\theta_i)$ is a Cartan decomposition. For $\lambda \in \theta_i$ we fix $X \in \mathfrak{g}_\lambda$ with $-B(X, \theta X) = 1$. Then $[\theta X, X] = Q_\lambda \in \mathfrak{g}(\theta_i)$. Let $\mathfrak{a}_i$ denote the orthogonal complement (relative to B_θ) of $\Sigma_{\lambda\in\theta_i} \mathbf{R} \cdot Q_\lambda$ in $\mathfrak{a}$. For $x = xE_{12} \in \mathfrak{g}_{e_1-e_2}$, $Q_\lambda = D(+x^2, -x^2, 0, \cdots, 0)$ and $\operatorname{tr}(D(a_1, \cdots, a_n)Q_\lambda) = x^2(a_1 - a_2)$. Since $B(X, Y)$ is a multiple of $\operatorname{tr}(XY)$, we have that $\mathfrak{a}_1 = D(a_1, a_1, a_3, \cdots, a_n)$. Similarly, we have for $k \leqslant t$, $\mathfrak{a}_k = D(a_1, a_1, a_2, a_2, \cdots, a_k, a_k, a_{2k+1}, \cdots, a_n)$. Now set $\mathfrak{n}_i = \Sigma_{\lambda\in\Delta^+-\theta_i} \mathfrak{g}_\lambda$, $\mathfrak{v}_i = \theta(\mathfrak{n}_i)$, and

$$\mathfrak{p}_i = \mathfrak{p}_0 + \mathfrak{n}^-(\theta_i) = \mathfrak{n}^+(\theta_i) + \mathfrak{n}^-(\theta_i) + \sum_{\lambda\in\theta_i} \mathbf{R}Q_\lambda + \mathfrak{a}_i + \mathfrak{n}_i$$

($\mathfrak{p}_0 = LA(P_0)$). Denote by A_i, N_i, V_i the analytic subgroups of G corresponding respectively to $\mathfrak{a}_i, \mathfrak{n}_i, \mathfrak{v}_i$. Note that V_i has the following form

$$\begin{pmatrix} 1 & 0 & & & & & & & & & \\ 0 & 1 & & & & & & & & & \\ & & 1 & 0 & & & & & & & \\ & & 0 & 1 & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & 1 & 0 & & & & \\ & & & & & 0 & 1 & & & & \\ & & & & & & & 1 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & & 1 \end{pmatrix}$$

where there are i blocks of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let

$$\mathfrak{m}_i = \mathfrak{n}^+(\theta_i) + \mathfrak{n}^-(\theta_i) + \sum_{\lambda \in \theta_i} \mathbf{R}Q_\lambda,$$

i.e.,

$$\mathfrak{m}_k = \begin{pmatrix} X_1 & Y_1 & & & & & & & \\ Z_1 & -X_1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & X_k & Y_k & & & & \\ & & & Z_k & -X_k & & & & \\ & & & & & 0 & & & \\ & & & & & & \ddots & & \\ & & & & & & & & 0 \end{pmatrix}.$$

Let M_{θ_i} denote the analytic subgroup of G with Lie algebra $\mathfrak{m}_i$ and $M_{\theta_i}(K)$ denote the centralizer of $\mathfrak{a}_i$ in K. Then

$$M_{\theta_i} = \begin{pmatrix} SL_1(2, \mathbf{R}) & & & & & \\ & \ddots & & & & \\ & & SL_i(2, \mathbf{R}) & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

[the indices indicate copies of $SL(2, \mathbf{R})$] while $M_{\theta_i}(K)$ has the form

$$\begin{pmatrix} SO_1^{\pm}(2) & & & & & \\ & \ddots & & & & \\ & & SO_i^{\pm}(2) & & & \\ & & & \pm 1 & & \\ & & & & \ddots & \\ & & & & & \pm 1 \end{pmatrix}$$

where the determinant must, of course, be 1. Finally, let $M_i = M_{\theta_i} = M_{\theta_i}(K)M_{\theta_i}$. Then

$$M_i = \begin{pmatrix} SL_1^{\pm}(2, \mathbf{R}) & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & SL_i^{\pm}(2, \mathbf{R}) & & \\ & & & & & \pm 1 & \\ & & & & & & \cdot \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & & \pm 1 \end{pmatrix},$$

$P_i = M_iA_iN_i$ is a parabolic subgroup of G containing P_0 (with Langlands decomposition $M_iA_iN_i$), and P_iV_i is a dense open submanifold of G whose complement has zero Haar measure. Clearly each P_i for $i = 0, 1, \cdots, t$ is cuspidal and since $\dim A_i \neq A_j$, $i \neq j$, they are all nonassociate. Thus $P_0, \cdots, P_t$ is a complete set of nonassociate cuspidal parabolic subgroups of G. We shall denote the continuous series corresponding to P_i by $\{\pi_i(\sigma, \tau) = \mathrm{Ind}_{P_i}^G \sigma \times \tau\colon \sigma \in \hat{M}_{i_d}, \tau \in \hat{A}_i\}$.

Theorem 1 now says $\pi_0(\sigma_1, \tau_1) \otimes \pi_0(\sigma_2, \tau_2) \simeq \mathrm{Ind}_{MA}^G (\sigma_1\sigma_2 \times \tau_1\tau_2)$. Thus, by Theorem 2 and Anh's reciprocity formula, we need only find the multiplicity of $\sigma_1\sigma_2 \times \tau_1\tau_2$ in $(\pi)_{MA}$ for μ_G-almost all $\pi \in \hat{G}$, i.e., for almost all $\pi_i(\sigma, \tau)$, $i = 0, \cdots, t$. Since the various continuous series representations are all induced representations, we may apply the subgroup theorem.

(1) Let $\pi_0(\sigma, \tau)$ be a principal series representation. Since the action of MA on $MAN\backslash G$ on the left corresponds to the action of MA on V by inner automorphism, we may identify $MAN\backslash G/MA$ with V/MA. Let $\mathfrak{v} = LA(V)$ and $\mathfrak{v}^0 = \{\Sigma_{\alpha\in\Delta^-} X_\alpha\colon X_\alpha \neq 0 \text{ for each } \alpha \in \Delta^-\}$. Then $\mathfrak{v}^0$ is an open, dense, conull subset of $\mathfrak{v}$ invariant under the action of MA. Let $V^0 = \exp \mathfrak{v}^0$. Then V^0 is a dense, conull subset of V and the subgroups MAN, MA will be regularly related in G if we can show V^0/MA is countably separated. Since $V^0/MA \approx \mathfrak{v}^0/MA$ (recall that $\exp$ is a diffeomorphism and takes orbits to orbits), it suffices to show $\mathfrak{v}^0/MA$ is countably separated. This will follow from [3, p. 42] once we show that the orbits in $\mathfrak{v}^0/MA$ are locally closed [a subset Z of a topological space Y is locally closed if Z is the intersection of an open and closed set]. Since M is compact, it suffices to show that the orbits in $\mathfrak{v}^0/A$ are locally closed.

Now let $X = \Sigma X_\alpha \in \mathfrak{v}^0$, $Y = \Sigma Y_\alpha \in \mathfrak{v}^0$, and $\{\exp H_n\}$ be a sequence in A with $(\exp H_n) \cdot X \longrightarrow Y$. Then

$$(\exp H_n) \cdot X = \sum . e^{\alpha(H_n)} X_\alpha \longrightarrow \sum Y_\alpha$$

$$\Rightarrow e^{\alpha(H_n)} \longrightarrow Y_\alpha / X_\alpha > 0 \text{ for each } \alpha \in \Delta^-$$

$$\Rightarrow \alpha(H_n) \text{ converges for each } \alpha \in \Delta^-$$

$$\Rightarrow \beta(H_n) \text{ converges for each } \beta \in \mathfrak{a}^*$$

$$\Rightarrow \exists H \in \mathfrak{a} \text{ such that } \beta(H_n) \longrightarrow \beta(H) \text{ for all } \beta \in \mathfrak{a}^*.$$

Now $\exp H \in A$ and

$$(\exp H_n) \cdot X = \sum_{\alpha \in \Delta^-} e^{\alpha(H_n)} X_\alpha \longrightarrow \sum_{\alpha \in \Delta^-} e^{\alpha(H_n)} X_\alpha = (\exp H) \cdot X.$$

Thus

$$(\exp H) \cdot X = Y, \qquad Y \in \mathcal{O}_X = \{a \cdot X\colon\ a \in A\},$$

and $\mathcal{O}_X$ is *closed* in $\mathfrak{v}^0$.

We now show that the stability subgroup of any $X \in \mathfrak{v}^0$ under the action of MA equals $Z(G)$. Let $c = D(c_1, \cdots, c_n) \in MA$ and let

$$\begin{pmatrix} 1 & & & & & \\ X_{21} & 1 & & & & \\ X_{31} & X_{32} & 1 & & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ \cdot & \cdot & & & & \cdot \\ X_{n1} & X_{n2} & \cdots & X_{nn-1} & & 1 \end{pmatrix}$$

$$= (x_{21}, x_{31}, x_{32}, \cdots, x_{n1}, \cdots, x_{nn-1}) \in V.$$

Then

$$c \cdot (x_{21}, \cdots, x_{ij}, \cdots, x_{nn-1}) = c(x_{21}, \cdots, x_{ij}, \cdots, x_{nn-1})c^{-1}$$
$$= (c_2 c_1^{-1} x_{21}, \cdots, c_i c_j^{-1} x_{ij}, \cdots, c_n c_{n-1}^{-1} x_{nn-1}).$$

Now let $x = xE_{ij} \in \mathfrak{g}_{e_j - e_i}$, $j < i$. Then $\exp X = I + xE_{ij}$, and $\exp(c_i c_j^{-1} X) = c \cdot \exp(X) = \exp \operatorname{Ad}(c)X = \exp(c \cdot X)$. Thus $c \cdot X = c_i c_j^{-1} xE_{ij}$ and MA acts in one nonzero orbit on each root space. Now suppose $X = \Sigma_{j<i} x_{ij} E_{ij} \in \mathfrak{v}^0$ and $c = D(c_1, \cdots, c_n) \in MA$ with $c \cdot X = X$. Then $c \cdot X = \Sigma_{j<i} c_i c_j^{-1} x_{ij} E_{ij} = \Sigma x_{ij} E_{ij}$ and $x_{ij} \neq 0$, $j < i$, implies that $c_{n-1} c_{n-2}^{-1} = c_n c_1^{-1} = \cdots = c_n c_{n-2}^{-1} = 1$, i.e., $c_n = c_1 = c_2 = \cdots = c_{n-2} = c_{n-1}$. Since $1 = \Pi_{i=1}^n c_i = c_n^n$, we have that $c \in Z(G)$. Thus

$$\{c \in MA:\ c \cdot X = X \ \text{ for } \ X \in \mathfrak{v}^0\}$$
$$= \{c \in MA:\ c \cdot v = v \ \text{ for } \ v \in V^0 = \exp \mathfrak{v}^0\} = Z(G).$$

Now by Mackey's subgroup theorem, we have

$$(\mathrm{Ind}_{MAN}^G \ \sigma \times \tau)_{MA} \simeq \int_{V^0/MA}^{\oplus} \mathrm{Ind}_{Z(G)}^{MA}\ (\sigma)_{Z(G)}\, d\nu(v)$$
$$\simeq \infty(\mathrm{Ind}_{Z(G)}^{M}\ (\sigma)_{Z(G)} \times \rho)$$

where ρ denotes the regular representation of A and ν is any admissible measure on V/MA.

(2) We now find $(\mathrm{Ind}_{P_i}^G \ \sigma \times \tau)_{MA}$ for $i = 1, \cdots, t$. Since the action of MA on $P_i\backslash G$ corresponds to the action of MA on V_i by inner automorphism, we may identify $P_i\backslash G/MA$ with V_i/MA. Let $V_i^0 = V_i \cap V^0$. Then V_i^0 is a dense, conull subset of V_i such that the stability subgroup at any $v \in V_i^0$ is equal to $Z(G)$ (same proof as in (1)). By an argument similar to that in (1), we have that V_i^0/MA is countably separated. If we now let ν_i denote any admissible measure on V_i/MA, we have by Mackey's subgroup theorem

$$(\mathrm{Ind}_{P_i}^G \ \sigma \times \tau)_{MA} \simeq \int_{V_i/MA}^{\oplus} \mathrm{Ind}_{Z(G)}^{MA}\ (\sigma)_{Z(G)}\, d\nu_i(v).$$

For

$$n = 3, \quad t = 1, \quad V_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}, \quad Z(G) = \{e\},$$

there is one nonzero orbit in V_1^0/MA, $M_1 = \mathrm{SL}^{\pm}(2, \mathbf{R})$, and for $\sigma \in \hat{M}_{1_d}$ we have $(\sigma)_{Z(G)} = \infty \cdot 1$. Thus

$$(\mathrm{Ind}_{P_1}^G \ \sigma \times \tau)_{MA} \simeq \infty \, \mathrm{Ind}_{\{e\}}^{MA} 1 \simeq \infty \ {}^{MA}R.$$

For $n > 3$, $\dim (V_i/MA) \geqslant 1$ and so for $i = 1, \cdots, t$

$$(\mathrm{Ind}_{P_i}^G \ \sigma \times \tau)_{MA} \simeq \infty(\mathrm{Ind}_{Z(G)}^{M}\ (\sigma)_{Z(G)} \times \rho).$$

So for n odd and $i = 0, 1, \cdots, t$ we have $(\mathrm{Ind}_{P_i}^G \ \sigma \times \tau)_{MA} \simeq \infty \ {}^{MA}R$. For n even, we let

$$T^+ = \sum_{\sigma \in \hat{M}, (\sigma)_{Z(G)} = \chi^+} \sigma \quad \text{and} \quad T^- = \sum_{\sigma \in \hat{M}, (\sigma)_{Z(G)} = \chi^-} \sigma.$$

Then for $i = 0, 1, \cdots, t$

$$(\mathrm{Ind}_{P_i}^G \ \sigma \times \tau)_{MA} \simeq \begin{cases} \infty(T^+ \times \rho) & \text{if } (\sigma)_{Z(G)} \simeq \infty\chi^+, \\ \infty(T^- \times \rho) & \text{if } (\sigma)_{Z(G)} \simeq \infty\chi^-. \end{cases}$$

The following theorem is now clear.

THEOREM. *Let* $\pi_0(\sigma_1, \tau_1)$, $\pi_0(\sigma_2, \tau_2)$ *be two principal series representations for* $G = \mathrm{SL}(n, \mathbf{R})$ *for* $n \geqslant 3$. *Then if* n *is odd*, $\pi_0(\sigma_1, \tau_1) \otimes \pi_0(\sigma_2, \tau_2) \simeq {}^G R$, *while if* n *is even*

$$\pi_0(\sigma_1, \tau_1) \otimes \pi_0(\sigma_2, \tau_2) \simeq \begin{cases} \int_{\hat{G}_+}^{\oplus} \infty \pi \, d\mu_G(\pi) & \text{if } (\sigma_1\sigma_2)_{Z(G)} = \chi^+, \\ \int_{\hat{G}_-}^{\oplus} \infty \pi \, d\mu_G(\pi) & \text{if } (\sigma_1\sigma_2)_{Z(G)} = \chi^-, \end{cases}$$

where $\hat{G}_+ = \{\pi \in \hat{G}: (\pi)_{Z(G)} = \infty\chi^+\}$, $\hat{G}_- = \{\pi \in \hat{G}: (\pi)_{Z(G)} = \infty\chi^-\}$.

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